MATHEMATICS – IB

IPE STUDY MATERIAL

S.No.	Topic Name with Weightage	7M LAQ	4M SAQ	2M VSAQ
1	LOCUS	-	-	4
2	TRANSFORMATION OF AXES	-	-	4
3	THE STRAIGHT LINE	7	4	2+2
4	PAIR OF STRAIGHT LINES	7 + 7	-	-
5	THREE DIMNESIONAL COORDINATES	-	-	2
6	DIRECTION COSINES AND DIRECTINRATIOS	7	-	-
7	THE PLANE	-	-	2
8	LIMITS AND CONTINUITY	-	4 OR 4	2 2 + 2
9	DIFFERENTIATION	7 OR 7	4	2 + 2 + 2
10	APPLICATION OF DERIVATIONS			
10.1	ERRORS AND APPROXIMATIONS	-	-	2
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1. LOCUS

Definition : The set of all points which satisfies the given geometrical conditions (properties) is called a locus.

From the definition of the Locus, it follows that

I Every point satisfying the given condition is a point of the locus.

II Every point of the locus satisfies the given condition.

An equation of a locus is an algebraic description of the locus. This can be obtained in the following way.

i. Consider a point $P(x, y)$ on the locus.

ii. Write the geometric condition(s) to be satisfies by P in terms of an equation or inequation in symbols.

iii. Apply the proper formula of coordinate geometric and translate the geometric condition in to algebraic equation.

iv. Simplify the equation so that it is free from radicals.

v. Verify that of $Q(x_1,\ y_1)$ satisfies the equation, then Q satisfies the geometric condition. The equation thus obtained is the required equation of locus.

1. LOCUS

LEVEL- I

1. If the distance from P to the points (2, 3) and (2, -3) are in the ration 2 : 3 then find the equation of Locus of P

Sol: Let $P(x_1,\ y_1)$ be a point on the locus

Given A(2, 3)   B(2, -3)

Given condition is PA : PB = 2 : 3

⇒ 3 PA = 2PB ⇒ $9PA^2 = 4PB^2$

$\Rightarrow 9[(x_1-2)^2+(y_1-3)^2] = 4[(x_1-2)^2+(y_1+3)^2]$

$\Rightarrow 9[x_1^2+4-4x_1+y_1^2+9-6y_1] = 4[x_1^2+4-4x_1+y_1^2+9+6y_1]$

$\Rightarrow 9[x_1^2+y_1^2-4x_1-6y_1+13] = 4[x_1^2+y_1^2-4x_1+6y_1+13]$

$\Rightarrow x_1^2+y_1^2-20x_1-78y_1+65 = 0$

The locus of $P(x_1,\ y_1)$ is $x_1^2+y_1^2-20x_1-78y_1+65=0$

2. A(5, 3) and B(3, -2) are two fixed points. Find the equation of P, so that area of ΔPAB is 9 Sq. units.

Sol: A(5, 3), B(3, -2) are the given points

Let $P(x_1,\ y_1)$ be any point on the Locus

Given Condition is that the area of $\Delta PAB = 9$

$\Rightarrow \frac{1}{2}|x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)| = 9$

$\Rightarrow \frac{1}{2}|5(-2-y_1)+3(y_1-3)+x_1(3+2)| = 9$

$\Rightarrow |-10-5y_1+3y_1-9+3x_1+2x_2| = 18$

$\Rightarrow |5x_1-2y_1-19| = 18$

$\Rightarrow 5x_1-2y_1-19 = \pm 18$

$\Rightarrow (5x_1-2y_1-19) = 18$ or $5x_1-2y_1-19 = -18$

$\Rightarrow 5x_1-2y_1-37 = 0$ or $5x_1-2y_1-1=0$

∴ Locus of P is $(5x-2y-37)(5x-2y-1)=0$

$\Rightarrow 25x^2-20xy+4y^2-190x+76y+37=0$

3. A(2, 3) and B(-3, 4) are two given points. Find the equation of Locus of P, so that the area of the triangle PAB is 8.5.

Sol: A(2, 3), B(-3, 4) are the given points

Let $P(x_1,\ y_1)$ be any point on the Locus

Given Condition is that the area of $\Delta PAB = 8.5$

$\Rightarrow \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)| = 8.5$

$\Rightarrow \frac{1}{2}|x_1(3 - 4) + 2(4 - y_1) + (-3)(y_1 - 3)| = 8.5$

$\Rightarrow |-x_1 + 8 - 2y_1 - 3y_1 + 9| = 17$

$\Rightarrow |-x_1 - 5y_1 + 17| = 17$

$\Rightarrow -x_1 - 5y_1 + 17 = \ \pm 17$

$\Rightarrow (-x_1 - 5y_1 + 17) = 17$ or $-x_1 - 5y_1 + 17 = \ -17$

$\Rightarrow x_1 - 5y_1 = 0$ or $-x_1 - 5y_1 + 34 = 0$

$\therefore$ Locus of P is $(-x - 5y)(-x - 5y + 34) = 0$

$\Rightarrow x^2 + 10xy + 25y^2 - 34x - 170y = 0$

4. Find the Locus of P, of the line segment joining (2, 3) and (-1, 5) subtends a right angle at P.

Sol: Let $P(x_1,\ y_1)$ be a point on the Locus.

Let A(2, 3) B(-1, 5) be the given two points

Given condition is $\angle APB = 90^o$

$\Rightarrow$ (Slope of AP) × (Slope of BP) = -1

$\Rightarrow \left(\frac{y_1-3}{x_1-2}\right)\left(\frac{y_1-5}{x_1+1}\right) = -1$

$\Rightarrow y_1^2 - 8y_1 + 15 = -(x_1^2 - x_1 - 2)$

$\Rightarrow x_1^2 + y_1^2 - x_1 - 8y_1 + 13 = 0$

The Locus of $P(x_1,\ y_1)$ is $x^2 + y^2 - x - 8y + 13 = 0$

5. The ends of the hypotenuse of a right angled triangle are (0, 6) and (6, 0). Find the Locus of its third vertex.

Sol: Let $P(x_1,\ y_1)$ be a point on the Locus.

Let (0, 6) and (6, 0) be the given two points

Given condition is $\angle APB = 90^o$

⇒ (Slope of AP) × (Slope of BP) = -1

⇒ $\left(\frac{y_1-6}{x_1-0}\right)\left(\frac{y_1-0}{x_1-6}\right) = -1$

⇒ $y_1^2 - 6y_1 = -(x_1^2 - 6x_1)$

⇒ $x_1^2 + y_1^2 - 6x_1 - 6y_1 = 0$

The Locus of $P(x_1,\ y_1)$ is $x^2 + y^2 - 6x - 6y = 0$

6. A(1, 2) B(2, -3) C(-2, 3) are three points. A point P moves such that $PA^2 + PB^2 = 2PC^2$. Show that the equation to the Locus of P is $7x - 7y + 4 = 0$.

Sol: Let $P(x_1,\ y_1)$ be a point on the locus

Given three points are A(1, 2), B(2, -3), C(-2, 3)

Given geometric Condition is $PA^2 + PB^2 = 2PC^2$.

⇒ $(x_1 - 1)^2 + (y_1 - 2)^2 + (x_1 - 2)^2 + (y_1 + 3)^2 = 2[(x_1 + 2)^2 + (y_1 - 3)^2]$

⇒ $x_1^2 + 1 - 2x_1 + y_1^2 + 4 - 4y_1 + x_1^2 + 4 - 4x_1 + y_1^2 + 9 + 6y_1 =$

$2[x_1^2 + 4 + 4x_1 + y_1^2 + 9 - 6y_1]$

⇒ $2x_1^2 + 2y_1^2 - 6x_1 + 2y_1 + 18 = 2x_1^2 + 2y_1^2 + 8x_1 - 12y_1 + 26$

⇒ $14x_1 - 14y_1 + 8 = 0 \Rightarrow 7x_1 - 7y_1 + 4 = 0$

∴ The Locus of $P(x_1,\ y_1)$ is $7x - 7y + 4 = 0$

LEVEL – II

1. Find the equation of the Locus of P, if A = (2, 3), B = (2, -3) and PA + PB = 8.

Sol: Given points are A(2, 3) and B(2, -3)

Let $P(x_1,\ y_1)$ be the point on the Locus

Given Condition is PA + PB = 8 -------------- (1)

$PA^2 - PB^2 = [(x_1-2)^2+(y_1-3)^2]-[(x_1-2)^2+(y_1+3)^2]$

$= (x_1-2)^2+(y_1-3)^2-(x_1-2)^2-(y_1+3)^2$

$= y_1^2+9-6y_1-y_1^2-9-6y_1 = 12y_1$

$\Rightarrow PA^2 - PB^2 = 12y_1$

$\Rightarrow (PA + PB)(PA - PB) = 12y_1$

$\Rightarrow 8\ (PA - PB) = 12y_1 \Rightarrow PA - PB = \frac{-12y_1}{8} = \frac{-3y_1}{2}$ -------- (2)

Adding (1) & (2)

$2PA = 8 - \frac{3y_1}{2} = \frac{16-3y_1}{2}$

$\Rightarrow 4PA = 16-3y_1$

$\Rightarrow 16PA^2 = (16-3y_1)^2$

$\Rightarrow 16\ [(x_1-2)^2+(y_1-3)^2] = 256+9y_1^2-96y_1$

$\Rightarrow 16[x_1^2+4-4x_1+y_1^2+9-6y_1] = 256+9y_1^2-96y_1$

$\Rightarrow 16x_1^2+7y_1^2-64x_1-48=0$

$\Rightarrow 16x_1^2-64x_1+7y_1^2=48$

$\Rightarrow 16x_1^2-64x_1+64+7y_1^2=48+64$

$\Rightarrow 16(x_1^2-4x_1+4)+7y_1^2=112$

$\Rightarrow 16(x_1-2)^2+7y_1^2=112$

$\Rightarrow \frac{(x_1-2)^2}{7}+\frac{y_1^2}{16}=1$

2. Find the equation of Locus of P, if A= (4, 0), B = (-4, 0) and $|PA - PB| = 4$.

Sol: Let $P(x_1,\ y_1)$ be the point on the Locus

Given points are A(4, 0) and B(-4, 0)

Given geometric condition is $|PA - PB| = 4$

$\Rightarrow PA - PB = \pm 4 \Rightarrow PA = \pm 4 + PB$

Squaring on both sides $\Rightarrow (PA)^2 = (\pm 4 + PB)^2$

$\Rightarrow (PA)^2 = 16 + (PB)^2 \pm 8PB$

$\Rightarrow (PA)^2 - (PB)^2 = 16 \pm 8PB$

$\Rightarrow [(x_1 - 4)^2 + (y_1 - 0)^2] - [(x_1 + 4)^2 + (y_1 - 0)^2] = 16 \pm 8PB$

$\Rightarrow x_1^2 - 8x_1 + 16 + y_1^2 - x_1^2 - 8x_1 - 16 - y_1^2 = 16 \pm 8PB$

$\Rightarrow -16x_1 = 8(2 \pm PB)$

$\Rightarrow -2x_1 = 2 \pm PB$

$\Rightarrow -2x_1 - 2 = \pm PB$

Squaring on both sides

$\Rightarrow [-(2x_1 + 2)]^2 = (\pm PB)^2$

$\Rightarrow 4x_1^2 + 8x_1 + 4 = (PB)^2$

$\Rightarrow 4x_1^2 + 8x_1 + 4 = (x_1 + 4)^2 + (y_1 - 0)^2$

$\Rightarrow 4x_1^2 + 8x_1 + 4 = x_1^2 + 8x_1 + 16 + y_1^2$

$\Rightarrow 4x_1^2 + 4 - x_1^2 - 16 - y_1^2 = 0$

$\Rightarrow 3x_1^2 - y_1^2 - 12 = 0$

$\Rightarrow 3x_1^2 - y_1^2 = 12 \Rightarrow \frac{x_1^2}{4} - \frac{y_1^2}{12} = 1$

$\therefore$ The Locus of $P(x_1,\ y_1)$ is $3x^2 - y^2 = 12$ (or) $\frac{x^2}{4} - \frac{y^2}{12} = 1$

3. Find the equation of Locus of a point, the difference of whose distance from (-6, 0) and (5, 0) is 8.

Sol: Let $P(x_1,\ y_1)$ be the point on the Locus

Given points are A(-6, 0) and B(5, 0)

Given geometric condition is $|PA - PB| = 8$

$\Rightarrow PA - PB = \pm 8 \Rightarrow PA = \pm 8 + PB$

Squaring on both sides $\Rightarrow (PA)^2 = (\pm 8 + PB)^2$

$\Rightarrow (PA)^2 = 64 + (PB)^2 \pm 16PB$

$\Rightarrow (PA)^2 - (PB)^2 = 64 \pm 16PB$

$\Rightarrow [(x_1 + 5)^2 + (y_1 - 0)^2] - [(x_1 - 5)^2 + (y_1 - 0)^2] = 64 \pm 16PB$

$\Rightarrow x_1^2 + 10x_1 + 25 + y_1^2 - x_1^2 + 10x_1 - 25 - y_1^2 = 64 \pm 16PB$

$\Rightarrow 20x_1 = 4(16 \pm 4PB)$

$\Rightarrow 5x_1 = 16 \pm 4PB$

$\Rightarrow 5x_1 - 16 = \pm 4PB$

Squaring on both sides

$\Rightarrow (5x_1 - 16)^2 = 16\,(PB)^2$

$\Rightarrow 25x_1^2 - 160x_1 + 256 = 16[(x_1 - 5)^2 + (y_1 - 0)^2]$

$\Rightarrow 25x_1^2 - 160x_1 + 256 = 16(x_1^2 - 10x_1 + 25 + y_1^2)$

$\Rightarrow 4x_1^2 + 8x_1 + 4 = x_1^2 + 8x_1 + 16 + y_1^2$

$\Rightarrow 25x_1^2 - 160x_1 + 256 - 16x_1^2 + 160x_1 - 400 - 16y_1^2 = 0$

$\Rightarrow 9x_1^2 - 16y_1^2 - 144 = 0$

$\Rightarrow 9x_1^2 - 16y_1^2 = 144$

$\therefore$ The Locus of $P(x_1,\ y_1)$ is $9x^2 - 16y^2 = 144 \Rightarrow \frac{9x^2}{144} - \frac{16y^2}{144} = 1$

$\Rightarrow \frac{x^2}{16} - \frac{y^2}{9} = 1$

4. Find the equation of the Locus of a point, the sum of whose distance from (0, 2) and (0, -2) is 6.

Sol: Let $P(x_1,\ y_1)$ be the point on the Locus

Given points are A(0,2) and B(0, -2)

Given geometric condition is $PA + PB = 6$

$\Rightarrow PA = 6 - PB$

Squaring on both sides $\Rightarrow (PA)^2 = (6 - PB)^2$

$\Rightarrow (PA)^2 = 36 + (PB)^2 - 12PB$

$\Rightarrow (PA)^2 - (PB)^2 = 36 - 12PB$

$\Rightarrow [(x_1 - 0)^2 + (y_1 - 2)^2] - [(x_1 - 0)^2 + (y_1 + 2)^2] = 36 - 12PB$

$\Rightarrow x_1^2 + y_1^2 + 4 - 4y_1 - x_1^2 - y_1^2 - 4y_1 - 4 = 36 - 12PB$

$\Rightarrow -8y_1 = 12(3 - PB)$

$\Rightarrow --2y_1 = 3(3 - PB) \Rightarrow -2y_1 = 9 - 3PB$

$\Rightarrow -2y_1 - 9 = \ 3PB$

Squaring on both sides

$\Rightarrow 4y_1^2 + 36y_1 + 81 = 9[(x_1 - 0)^2 + (y_1 + 2)^2]$

$\Rightarrow\ 4y_1^2 + 36y_1 + 81 = 9(x_1^2 + y_1^2 + 4y_1 + 4)$

$\Rightarrow\ 4y_1^2 + 36y_1 + 81 - 9x_1^2 - 9y_1^2 - 36y_1 - 36 = 0$

$\Rightarrow\ -9x_1^2 - 5y_1^2 + 45 = 0$

$\Rightarrow\ 9x_1^2 + 5y_1^2 - 45 = 0$

$\Rightarrow 9x_1^2 + 5y_1^2 = 45$

$\therefore$ The Locus of $P(x_1,\ y_1)$ is $9x^2 + 5y^2 = 45\ \Rightarrow\ \frac{9x^2}{45} + \frac{5y^2}{45} = 1$

$$\Rightarrow \frac{x^2}{5} + \frac{y^2}{9} = 1$$

2. TRANSFORMATION OF AXES

- The axes can be transformed or changed usually in the following ways.
  I. Translation of axes
  II. Rotation of axes
  III. Translation and rotation of axes
- Translation of Axes: - If the origin is shifted to another point without changing the direction of the axes then the transformation is called Translation of axes.

  If the Coordinates (x, y) of a point are transformed to (X, Y) when the origin is shifted to (h, k) then $x = X + h \;\; y = Y + k$

  If the origin (0, 0) is shifted to (h, k) by the translation of axes then

  (i) the Coordinates (x, y) of a point P are transformed as $(x - h, y - k)$ and

  (ii) the equation $f(x, y)$ of the curve is transformed as $f(X + h,\ Y + k) = 0$.
- The points to which the origin is to be shifted by the translation of axes so as to remove the first degree terms from the equation.

  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ where $h^2 \neq ab$ is $\left(\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2}\right)$
- The point to which the origin has to be shifted $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is $\left(\frac{-g}{a}, \frac{-f}{b}\right)$
- Rotation of Axes: - The transformation obtained, by rotating both the coordinate axes in the plane by an equal angle, without changing the position of the origin is called a Rotation of axes.
- If the coordinates (x, y) of a points are transformed to (X, Y) when the axes are rotated through an angle θ about the origin then

  $x = Xcos\theta - Ysin\theta,\ y = Xsin\theta + Ycos\theta$
- If the axes are rotated through an angle θ then the equation $f(x, y) = 0$ of a curve is transformed $f(X\,cos\theta - Ysin\theta, Xsin\theta + Ycos\theta) = 0$
- If $f(X, Y) = 0$ is the transformed equation of a curve when the axes are rotated through an angle θ then the original equation of the curve is

  $f(x\,cos\theta + y\,sin\theta, -xsin\theta + ycos\theta) = 0$
- The angle of rotation of the axes to eliminate xy term in the equation

  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is $\frac{1}{2}tan^{-1}\left(\frac{2h}{a-b}\right)$ is $(a \neq b)$ and $\frac{\pi}{4}$ if $a = b$.

LEVEL – I

1. When the origin is shifted to (-1, 2) by the translation of axes, find the transformed equal of $2x^2 + y^2 - 4x + 4y = 0$

Sol: Let (x, y) be the old Coordinator

Let (X, Y) be the new coordinator after shifting the origin to (h, k) = (-1, 2) then

$x = X + h = X - 1 \quad y = Y + k = Y + 2$

Given equation $2x^2 + y^2 - 4x + 4y = 0$

Transformed Equation is $2(X - 1)^2 + (Y + 2)^2 - 4(X - 1) + 4(Y + 2) = 0$

$\Rightarrow 2[X^2 - 2X + 1] + Y^2 + 4Y + 4 - 4X + 4 + 4Y + 8 = 0$

$\Rightarrow 2X^2 - 4X + 2 + Y^2 + 4Y \pm 4X + 4 + 4Y + 8 = 0$

$\Rightarrow 2X^2 + Y^2 - 8X + 8Y + 18 = 0$

$\therefore$ The Transformed equation is $2x^2 + y^2 - 8x + 8y + 18 = 0$

2. If the transformed equation of a curve $x^2 + +3xy - 2y^2 + 17x - 7y - 11 = 0$ when the origin is shifted to the point (2, 3) then find the original equation of the curve.

Sol: Given the equation is $X^2 + +3XY - 2Y^2 + 17X - 7Y - 11 = 0$

(h, k) = (2, 3) $X = x - h = x - 2$, $Y = y - k = y - 3$

$(x - 2)^2 + +3(x - 2)(y - 3) - 2(y - 3)^2 + 17(x - 2) - 7(y - 3) - 11 = 0$

$\Rightarrow x^2 - 4x + 3xy - 9x - 6y + 18 - 2y^2 + 12y - 18 + 17x = 34 - 7y + 21 - 11 = 0$

$\Rightarrow x^2 - 2y^2 + 3xy + 4x - y - 20 = 0$

$\therefore$ The Transformed equation is $x^2 - 2y^2 + 3xy + 4x - y - 20 = 0$

3. When the axes are rotated through an angle $\frac{\pi}{4}$, find the transformed equation of $3x^2 + 10xy + 3y^2 = 9$.

Sol: Given equation is $3x^2 + 10xy + 3y^2 = 9$ ---------- (1)

Angle of rotation $\theta = \frac{\pi}{4}$

Let (X, Y) be the new coordinates of (x, y) then

$x = Xcos\theta - Ysin\theta = Xcos\frac{\pi}{4} - Ysin\frac{\pi}{4} = \frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}} = \frac{X-Y}{\sqrt{2}}$ -2

$y = Xsin\theta + Ycos\theta = Xsin\frac{\pi}{4} + Ycos\frac{\pi}{4} = \frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}} = \frac{X+Y}{\sqrt{2}}$

∴ Transformed equation of (1) is

$3\left(\frac{X-Y}{\sqrt{2}}\right)^2 + 10\left(\frac{X-Y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right) + 3\left(\frac{X+Y}{\sqrt{2}}\right)^2 = 9$

$\Rightarrow \frac{3(X^2+Y^2-2XY)}{2} + \frac{10(X^2-Y^2)}{2} + \frac{3(X^2+Y^2+2XY)}{2} - 9$

$\Rightarrow 3X^2 + 3Y^2 - 6XY + 10X^2 - 10Y^2 + 3X^2 + 3Y^2 + 6XY = 18(\quad)$

$\Rightarrow 16X^2 - 4Y^2 = 18 \Rightarrow 8X^2 - 2Y^2 = 9$

∴ The Transformed equation is $8x^2 - 2y^2 = 9$

4. When the axes are rotated through an angle α, find the transformed equation of $xcos\alpha + ysin\alpha = P$.

Sol: Given equation is $x = cos\alpha + ysin\alpha = P$ --------(1)

$\theta = \alpha$ $\quad x = Xcos\alpha - Ysin\alpha \quad y = Xsin\alpha + Ycos\alpha$

Transformed equation of (1) is

$(X\ cos\alpha - Y\ sin\alpha)cos\alpha + (X\ sin\alpha + Y\ cos\alpha)sin\alpha = P$

$\Rightarrow X\ cos^2\alpha - Ysin\alpha\ cos\alpha + Xsin^2\alpha + Ycos\alpha\ sin\alpha = P$

$\Rightarrow X(cos^2\alpha + sin^2\alpha) = P \Rightarrow \mathrm{X} = \mathrm{P}$

∴ The transformed equation is $x = P$

5. When the axes are rotated through an angle $\frac{\pi}{6}$, find the transformed equation of $x^2 + 2\sqrt{3}xy - y^2 = 2a^2$.

Sol: Given equation is $x^2 + 2\sqrt{3}xy - y^2 = 2a^2$ --------- (1)

$\theta = \frac{\pi}{6}$

Let (X, Y) be the new coordinates of (x, y) then

$x = Xcos\theta - Ysin\theta = Xcos\frac{\pi}{6} - Ysin\frac{\pi}{6} = X\left(\frac{\sqrt{3}}{2}\right) - Y\left(\frac{1}{2}\right) = \frac{\sqrt{3}X-Y}{2}$ -2

$y = Xsin\theta + Ycos\theta = Xsin\frac{\pi}{6} + Ycos\frac{\pi}{6} = X\left(\frac{1}{2}\right) + Y\left(\frac{\sqrt{3}}{2}\right) = \frac{X+\sqrt{3}Y}{2}$

∴ Transformed equation of (1) is

$\left(\frac{\sqrt{3}X-Y}{2}\right)^2 + 2\sqrt{3}\left(\frac{\sqrt{3}X-Y}{2}\right)\left(\frac{X+\sqrt{3}Y}{2}\right) - \left(\frac{X+\sqrt{3}Y}{2}\right)^2 = 2a^2$

$\Rightarrow \frac{3X^2+Y^2-2\sqrt{3}XY}{4} + \frac{2\sqrt{3}(\sqrt{3}X^2+3XY-XY-\sqrt{3}Y^2)}{4} + \frac{(X^2+3Y^2+2\sqrt{3}XY)}{4} = 2a^2$

$\Rightarrow 3X^2 + Y^2 - 2\sqrt{3}XY + 6X^2 + 4\sqrt{3}XY - 6Y^2 - X^2 - 2\sqrt{3}XY + 3Y^2 = 8a^2$

$\Rightarrow 8X^2 - 8Y^2 = 8a^2 \Rightarrow X^2 - Y^2 = a^2$

∴ The Transformed equation is $x^2 - y^2 = a^2$

6. When the axes are rotated through an angle 45°, the transformed equation of a curve is
   $17x^2 - 16xy + 17y^2 = 225$. Find the original equation of the curve.

Sol: Angle of rotation = 45°

$X = xcos\theta + ysin\theta = xcos45^o + ysin45^o = \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} = \frac{x+y}{\sqrt{2}}$ -2

$Y = -xsin\theta + ycos\theta = -xsin45^o + ycos45^o = \frac{-x}{\sqrt{2}} + \frac{y}{\sqrt{2}} = \frac{-x+y}{\sqrt{2}} = \frac{y-x}{\sqrt{\ }2}$

Given equation is $17X^2 - 16XY + 17Y^2 = 225$

$17\left(\frac{x+y}{\sqrt{2}}\right)^2 - 16\left(\frac{x+y}{\sqrt{2}}\right)\left(\frac{-x+y}{\sqrt{2}}\right) - \left(\frac{-x+y}{\sqrt{2}}\right)^2 = 225$

$\Rightarrow \frac{17(x^2+y^2+2xy)}{2} - \frac{16(y^2-x^2)}{2} + \frac{17(x^2+y^2-2xy)}{2} = 225$

$\Rightarrow 17x^2 + 17y^2 + 34xy - 16y^2 + 16x^2 + 17x^2 + 17y^2 - 34xy = 450$

$\Rightarrow 50x^2 + 18y^2 = 450 \Rightarrow 25x^2 + 9y^2 = 225$

∴ The Original equation is $25x^2 + 9y^2 = 225$

LEVEL – II

1. Show that the axes are to be rotated through an angle of $\frac{1}{2}tan^{-1}\left(\frac{2h}{a-b}\right)$ so as to remove the xy term from the equation $ax^2 + 2hxy + by^2 = 0$ if $a \neq b$ and through the angle $\frac{\pi}{4}$ if $a = b$

Sol: If the axes are rotated through an angle θ then

$x = Xcos\theta - Ysin\theta, \quad y = Xsin\theta + Ycos\theta$

Therefore the given equation transforms as

$a(Xcos\theta - Ysin\theta)^2 + 2h(Xcos\theta - Ysin\theta)(Xsin\theta + Ycos\theta) + b(Xsin\theta + Ycos\theta)^2 = 0$

to remove XY term from the equation, we have to equate the coefficient of XY term to Zero.

So, $(b - a)sin\theta\, cos\theta + h(cos^2\theta - sin^2\theta) = 0$

i.e., $hcos2\theta = \frac{a-b}{2}sin2\theta$

i.e., $tan2\theta = \frac{2h}{a-b}$ is $a \neq b$

and $hcos2\theta = 0$, is $a = b$

$\therefore\ \theta = \frac{1}{2}tan^{-1}\left(\frac{2h}{a-b}\right)$, if $a \neq b$ and

$\theta = \frac{\pi}{4}$ if $a = b$

2. When the origin is shifted to (-2, -3) and the axes are rotated through an angle 45° find the transformed equation of $2x^2 + 4xy - 5y^2 + 20x - 22y - 14 = 0$.

Sol: Here (h, k) = (-2, -3), h = -2, k = -3, $\theta = 45^o$

Let (X, Y) be the new coordinates of any point (x, y) in the plane after the transformation.

Then, $x = Xcos\theta - Ysin\theta + h = Xcos45^o - Ysin45^o - 2 = X\left(\frac{1}{\sqrt{2}}\right) - Y\left(\frac{1}{\sqrt{2}}\right) - 2$

$x = \left(\frac{X-Y}{\sqrt{2}}\right) - 2$

$y = Xsin\theta + Ycos\theta + k = Xsin45^o + Ycos45^o32 = X\left(\frac{1}{\sqrt{2}}\right) + Y\left(\frac{1}{\sqrt{2}}\right) - 3$

$y = \left(\frac{X+Y}{\sqrt{2}}\right) - 3$

On substituting these values in the given equation , we get

$2\left\{\left(\frac{X-Y}{\sqrt{2}}\right) - 2\right\}^2 + 4\left\{\left(\frac{X-Y}{\sqrt{2}}\right) - 2\right\}\left\{\left(\frac{X+Y}{\sqrt{2}} - 3\right)\right\} - 5\left\{\left(\frac{X+Y}{\sqrt{2}}\right) - 3\right\}^2 + 20\left\{\left(\frac{X-Y}{\sqrt{2}}\right) - 2\right\} -$

$22\left\{\left(\frac{X+Y}{\sqrt{2}}\right) - 3\right\} - 14 = 0$

$\Rightarrow 2\left[\frac{(X-Y)^2}{2} + 4 - 4\frac{(X-Y)}{\sqrt{2}}\right] + 4\left[\frac{X^2-Y^2}{2} - \frac{3(X-Y)}{\sqrt{2}} - \frac{2(X-Y)}{\sqrt{2}}\right] - 5\left[\frac{(X+Y)^2}{2} + 9 - 6\left(\frac{X+Y}{\sqrt{2}}\right)\right]$

$+10\sqrt{2}(X-Y) - 40 - 11\sqrt{2}(X+Y) - 66 - 14 = 0$

$\Rightarrow (X-Y)^2 + 2(X^2 - Y^2) - \frac{5}{2}(X+Y)^2 - 1 = 0$

(grouping similar term and cancelling)

$\Rightarrow X^2 + Y^2 - 2XY + 2X^2 - 2Y^2 - \frac{5}{2}(X^2 + Y^2 + 2XY) - 1 = 0$

$\Rightarrow 2X^2 + 2Y^2 - 4XY + 4X^2 - 4Y^2 - 5X^2 - 5Y^2 - 10XY - 2 = 0$

$\Rightarrow X^2 - 7Y^2 - 14XY - 2 = 0$

Hence the transformed equation is $x^2 - 7y^2 - 14xy - 2 = 0$

3. THE STRAIGHT LINES

Key Points:

- Inclination of a line: If a line makes an angle θ $(0 \le \theta < \pi)$ with X-axis measured in positive direction 'θ' is called inclination of line.

  (i) If θ = 0°, then the line is parallel to X-axis.

  (ii) if θ = 90°, then the line $l = 0$ is perpendicular to X-axis.

- Slope of a Line:

  (i) If θ is the inclination of a line then tanθ is called slope of the line. Slope of a line is denoted by m, then m = tanθ.

  (ii) Slope of line passing through $(x_1, y_1), (x_2, y_2)$ is $\frac{y_2 - y_1}{x_2 - x_1}$.

  (iii) Slope of a line $y = mx + c$ is m

  (iv) Slope of a line $ax + by + c = 0$ is $\frac{-a}{b}$

  Note:

  (i) The slope of X-axis is 0 (when θ = 0)

  (ii) The slope of a horizontal line is 0 (when θ = 0)

  (iii) The slope of Y-axis is not defined (when θ = 90°)

  (iv) The slope of a vertical line is not defined.

  (v) If m > 0 then 0° < θ < 90°

  (vi) If m < 0 then 90° < θ < 180°

  (vii) If m = 0 then θ = 0

  (viii) If the slope of a line 'L'is m ≠ 0 then the slope of any line perpendicular to L = -1/m.

  (ix) If θ is on acute angle between two non vertical lines having slopes m_1 and m_2 then $tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$.

- General equation of Line: Every first degree equation in x and y represents a line. The equation of a line general form is $ax + by + c = 0$, $a, b, c \in R$, $a^2 + b^2 \neq 0$, having slope = $\frac{-c}{a}$

  x-intercept = $\frac{-c}{a}$, y-axis intercept = $\frac{-c}{b}$

  - The equation of a line parallel to $ax + by + c = 0$ is of the form $ax + by + k = 0$
  - The equation of a line perpendicular to $ax + by + c = 0$ is of the form $bx - ay + k = 0$.
  - The equation of x-axis is $y = 0$. The equation of any line parallel to x-axis is $y = k$.
  - The equation of y-axis is $x = 0$. The equation of any line parallel to y-axis is $x = k$.

- Equation of a Straight line in Various forms:
  - Two point form: The equation of the line passing through (x_1, y_1) and (x_2, y_2) $x_1 \neq x_2$ is $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ or $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$
  - Point – slope form: The equation of the line passing through (x_1, y_1) and having slope 'm'is $y - y_1 = m(x - x_1)$
  - Slope -intercept form: Equation of the line having slope 'm' and y-intercept 'c'is $y = mx + c$
  - Intercept form: Suppose a line L makes intercepts of a and b on x and y axes respectively then its equation is $\frac{x}{a} + \frac{y}{b} = 1.$
  - Equation of a line passing through (x_1, y_1) nnd

    (i) Parallel to $ax + by + c = 0$ is $a(x - x_1) + b(y - y_1) = 0$

    (ii) Perpendicular to $ax + by + c = 0$ is $b(x - x_1) - a(y - y_1) = 0$
- Normal form: The equation of a straight line, whose distance from the origin is 'P'and this perpendicular makes an angle α with positive direction of X-axis measured Counter clock wise direction is $xcos\alpha + ysin\alpha = P.$
  - The normal form of a line $ax + by + c = 0$ is

    $\frac{(-a)}{\sqrt{a^2+b^2}}x + \frac{(-b)}{\sqrt{a^2+b^2}}y = \frac{c}{\sqrt{a^2+b^2}}$, if c > 0 and

    $\frac{a}{\sqrt{a^2+b^2}}x + \frac{b}{\sqrt{a^2+b^2}}y = \frac{-c}{\sqrt{a^2+b^2}}$, if c < 0
- Symmetric form and Parametric form:
  - The equation of a straight line passing through (x_1, y_1) and making an angle θ with the positive direction of x-axis is

    $\frac{x - x_1}{cos\theta} = \frac{y - y_1}{sin\theta}$ where $\theta \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$
  - The coordinates (x, y) of any point P on the line at a distance 'r' units away from the point $A(x_1, y_1)$ can be taken as $(x_1 + rcos\theta, y_1 + rsin\theta)$ or $(x_1 - rcos\theta, y_1 - rsin\theta)$
  - The equation $x_1 \pm rcos\theta,\ y_1 \pm rsin\theta$ are called parametric equation of a line with parameter 'r'of the line passing through the point (x_1, y_1) having inclination θ.=+
- Distances:
  - The perpendicular distance to the line $ax + by + c = 0.$
  - From the origin is $\frac{|c|}{\sqrt{a^2+b^2}}$
  - From the point (x_1, y_1) is $\frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$

- The distance between parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is $\frac{|c_1 - c_2|}{\sqrt{a^2+b^2}}$
- The ration in which the line $L \equiv ax + by + c = 0\ (ab \neq 0)$ divides the line segment AB joining points $A(x_1, y_1)\ B(x_2, y_2)$ is $-L_{11} = L_{22}$ where $L_{11} = ax_1 + by_1 + c = 0\ L_{22} = ax_2 + by_2 + c = 0.$
- The Points A, B lie on the same side or opposite side of the line L = 0 according as L_{11}, L_{22} have same sign or opposite sign that is $-L_{11} : L_{22} < 0$ or $-L_{11} : L_{22} > 0$.
- If (h, k) is the foot of the perpendicular from (x_1, y_1) to the line $ax + by + c = 0$, then $\frac{h - x_1}{2} = \frac{k - y_1}{2} = -\frac{(ax_1 + by_1 + c)}{a^2 + b^2}$
- If (h, k) is the image of the point (x_1, y_1) with respect to the line $ax + by + c = 0$, then $\frac{h-x_1}{2} = \frac{k-y_1}{2} = \frac{-2(ax_1+by_1+c)}{a^2+b^2}$.
- The point of intersection of lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is $\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$
- The angle between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is Θ where $(0 \leq \theta \leq \pi)$ then $cos\theta = \frac{a_1a_2+b_1b_2}{\sqrt{a_1^2+b_1^2}.\sqrt{a_2^2+b_2^2}}$, $sin\theta = \frac{a_1a_2-b_1b_2}{\sqrt{a_1^2+b_1^2}.\sqrt{a_2^2+b_2^2}}$ and $tan\theta = \frac{a_1b_2-a_2b_1}{a_1a_2+b_1b_2}$
- Lines are perpendicular $\Leftrightarrow a_1a_2 + b_1b_2 = 0$.
- Lines are parallel $\Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2}$
- If $a_1x + b_1y + c_1 = 0,\ a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ represent three lines no two of which are parallel, then a necessary and sufficient condition for these lines to be concurrent is $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

- Centroid of a Triangle: The line segment joining vertex and mid point of opposite side of triangle is called a median of the triangle. The point of concurrency of the mediam of a triangle is called the centroid of a triangle. $A(x_1, y_1)\ B(x_2, y_2)\ C(x_3, y_3)$ are the vertices of a triangle ABC then centroid = $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$
- Circumcentre of a triangle: The point of concurrency of perpendicular bisectors of the sides of a triangle is called the circum centre. Denoted by S.

Note: (i) The circle which passes through all vertices of triangle is called the circumcircle of the ΔABC. The centre of circumcircle is called the circumcenter and the radius of circumcircle is called the radius of the triangle. It is denoted by R.

(ii) If S is the circumcenter of ΔABC, then SA = SB = SC = R

(iii) in a right angled triangle, the circumcenter is mid point of hypotenuse.

- Incentre of Triangle: The internal bisector of the angles A, B, C of ΔABC are concurrent at I. The point of I is the incentre of ΔABC.

  If $A(x_1, y_1)$ $B(x_2, y_2)$ $C(x_3, y_3)$ are the vertices of a ΔABC and BC = A, CA = b, AB = C then Incentre

  $I = \left[\frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c}\right]$

VSAQ (2 Marks)

1. Find the slope of the line passing through the points $(-p, q)$, $(q, -p)$, $(pq \neq 0)$.

Sol: Slope of the line passing through (x_1, y_1) & (x_2, y_2) is

Slope (m) = $\frac{y_2-y_1}{x_2-x_1} = \frac{-p-q}{q-(-p)} = \frac{-(p+,q)}{q+p} = -1$. ∴ m = -1

2. Find the value of x, if the slope of the line passing through (2, 5) and (x, 3) is 2.

Sol: Slope of the line joining (2, 5) and (x, 3) is

Slope (m) = $\frac{y_2-y_1}{x_2-x_1} = \frac{3-5}{x-2} = 2 \Rightarrow \frac{-2}{x-2} = 2$

$\Rightarrow -2 = 2(x-2) \Rightarrow x - 2 = -1 \Rightarrow x = -1 + 2 \Rightarrow x = 1$

3. Find the value of P, if the straight lines $3x + 7y - 1 = 0$ and $7x - Py + 3 = 0$ are mutually perpendicular.

Sol: Slope of the line $3x + 7y - 1 = 0$ is $m_1 = \frac{-3}{7}$

Slope of the line $7x - Py + 3 = 0$ is $m_2 = \frac{-7}{-P} \Rightarrow m_2 = \frac{7}{P}$

Given that the lines are perpendicular

∴ $m_1 m_2 = -1 \Rightarrow \frac{-3}{7} . \frac{7}{P} = -1 \Rightarrow -3 = -P \Rightarrow P = 3$

4. Find the value of y, if the line joining (3, y) and (2, 7) is parallel to the line joining the points (-1, 4) and
(0, 6).

Sol: Given points are A(3, y) B(2, 7) C(1, 4) D(0, 6)

Slope of AB = $\frac{7-y}{2-3} = \frac{7-y}{(-1)} = y - 7$

Slope of CD = $\frac{4-6}{-1-0} = \frac{-2}{(-1)} = 2$

AB and CD are parallel ⇒ slopes are equal

$y - 7 = 2 \Rightarrow y = 9$

5. Find the equation of the straight line passing through (-4, 5) and cutting off equal and non zero intercept on the coordinate axes.

Sol: Equation of the straight line in the intercept form is $\frac{x}{a} + \frac{y}{b} = 1$.

Given equal and non zero intercepts ⇒ a = b

∴ equation of the line is $\frac{x}{a}+\frac{y}{b}=1 \Rightarrow x+y=a$

The line passing through (-4, 5) ⇒ $-4+5=a$ ⇒ a = 1

∴ equation of the required line is $x+y=1$.

6. Find the equation of the straight line passing through the point (-2, 4) and making intercepts.

Sol: whose sum is

Equation of the line in the intercept form is $\frac{x}{a}+\frac{y}{b}=1$

Given a + b = 0 ⇒ b = -a

∴ equation of the line is $\frac{x}{a}+\frac{y}{-a}=1 \Rightarrow x-y=a$

The line is passing through (-2, 4)

$\therefore\ -2-4=a \Rightarrow a=-6$

∴ equation of the require line is $x-y=-6 \Rightarrow x-y+6=0$

7. Find the angle made by the straight line $y=-\sqrt{3x}+3$ with the positive direction of the x-axis measured in the counter clockwise direction.

Sol: Equation of the given line is $y=-\sqrt{3x}+3$.

Suppose $'\alpha'$ is the angle made by this line with positive x-axis in the counter clockwise direction

$\Rightarrow \tan\alpha = -\sqrt{3} = tan\frac{2\pi}{3}$ hence $\alpha=\frac{2\pi}{3}$

8. Show that the points (-5, 1), (5, 5) (10, 7) are collinear and find the equation of the straight line containing them.

Sol: Given points are A(-5, 1) B(5, 5) C(10, 7)

Equation straight line passing through (x_1, y_1) & (x_2, y_2)

$$y-y_1=\left(\frac{y_2-y_1}{x_2-x_1}\right)(x-x_1)$$

Equation of the line passing through A(-5, 1) B(5, 5) is

$$y-1=\left(\frac{5-1}{5+5}\right)(x+5)$$

$\Rightarrow 10(y-1) = 4(x+5)$

$\Rightarrow 4x + 20 - 10y + 10 = 0$

$\Rightarrow 4x - 10y + 30 = 0 \Rightarrow$ 2x-5y+15=0 ---------- (1)

Substituting C(10, 7) in eq (1)

⇒ 2(10) – 5(7) + 15 = 0

= 20 – 35 + 15 = 0 ⇒ 35 – 35 = 0

Hence the given points are Collinear

∴ The equation of the line containing the given points is $2x - 5y + 15 = 0$

9. Find the equation of the straight line perpendicular to the line $5x - 3y + 1 = 0$ and passing through the point (4, -3).

Sol: Equation of the line $5x - 3y + 1 = 0$, slope = 5/3

Slope of the line perpendicular to the given line is = -3/5.

Equation of line passing through (4, -3) and having slope -3/5 is

$y + 3 = -\frac{3}{5}(x-4)$

$\Rightarrow 5y + 15 = -3x + 12 \;\; \Rightarrow 3x + 5y + 3 = 0$

10. Find the area of the triangle formed by $x - 4y + 2 = 0$ with the coordinate axes.

Sol: Given line is $x - 4y + 2 = 0$

Here a = 1 b = -4 c= 2

Area of triangle formed by $ax + by + c = 0$ is $\frac{c^2}{2|ab|}$.

$$= \frac{(2)^2}{2|1(-4)|} = \frac{4}{2\times 4} = \frac{1}{2}$$

∴ Area of the triangle = $\frac{1}{2}$ Sq. Units.

11. Find the ratio in which (i) the x-axis and (ii) the y-axis divide the line segment $\overline{AB}$ joining A(2, -3) and
B(3, -6).

Sol: (i) x-axis divides $\overline{AB}$ in the ratio $-y_1 : y_2 = -3 : 6 = -1 : 2$

(ii) y-axis divides $\overline{AB}$ in the ratio $-x_1 : x_2 = -2 : 3$

12. Find the perpendicular distance from the point (-3, 4) to the straight line $5x - 12y = 2$.

Sol: Perpendicular distance of the point (-3, 4) from the line $5x - 12y - 2 = 0$ is equal to

$$= \frac{|5(-3)-12(4)-2|}{\sqrt{5^2+12^2}} = \frac{|-15-48-21|}{\sqrt{169}} = \frac{65}{13} = 5$$

13. Find the distance between the parallel lines $3x - 4y = 12$, $3x - 4y = 7$.

Sol: equation of the lines are $3x - 4y - 12 = 0$ and $3x - 4y - 7 = 0$ have

$a = 3$ $b = -4$ $c_1 = 12$, $c_2 = -7$

Distance between the parallel lines $\frac{|c_1-c_2|}{\sqrt{a^2+b^2}}$

$$= \frac{|-12+7|}{\sqrt{9+16}} = \frac{|-5|}{\sqrt{25}} = \frac{5}{5} = 1$$

14. Find the distance between the parallel lines $5x - 3y - 4 = 0$, $10x - 6y - 9 = 0$.

Sol: the equation of the given straight lines can be taken as

$10x - 6y - 8 = 0$, $10x - 6y - 9 = 0$

Distance between the parallel lines = $\frac{-8+9}{\sqrt{100+36}} = \frac{1}{\sqrt{136}} = \frac{1}{2\sqrt{34}}$

15. Find the equation of the straight line parallel to the line $2x + 3y + 7 = 0$ and passing through the point (5, 4).

Sol: equation of the given line is $2x + 3y + 7 = 0$

Required equation $2x + 3y = k$ passing through the point (5, 4)

$\therefore$ 2(5) + 3(4) = k $\Rightarrow$ k = 10 +12 $\Rightarrow$ k = 22

$\therefore$ equation of the line parallel to $2x + 3y + 7 = 0$ is $2x + 3y = 22 \Rightarrow 2x + 3y - 22 = 0$

16. Find the equation of the straight line perpendicular to the line $5x - 3y + 1 = 0$ and passing through the point (4, -3).

Sol: equation of the given line is $5x - 3y + 1 = 0$ --------- (1)

Equation of the line perpendicular to (1) is of the form $3x + 5y + K = 0$ -------- (2)

If (2) passes through (4, -3) then

3(4) + 5(-3) +k = 0

⇒ 12 – 15 + k = 0 ⇒ k = 3

∴ required line equation is $3x + 5y + 3 = 0$.

17. Find the value P if the straight lines $3x + 7y - 1 = 0$ and $7x - Py + 3 = 0$ are mutually perpendicular.

Sol: Equation of the given lines are $3x + 7y - 1 = 0$, $7x - Py + 3 = 0$

These lines are perpendicular ⇒ $a_1a_2 + b_1b_2 = 0$ ⇒ 3(7) +7(-P) = 0 ⇒ 21 – 7P = 0

⇒ 7P = 21 ⇒ P = 3

18. Find the condition for the points (a, 0) (h, k) and (0, b) where $ab \neq 0$ to be collinear.

Sol: A(a, 0), B(h, k), c(0, b) are collinear

⇒ slope of AB = Slope of AC

⇒ $\frac{k-0}{h-a} = \frac{-b}{a} \Rightarrow ak = -bh + ab$

⇒ $ak + bh = ab \Rightarrow \frac{ak}{ab} + \frac{bh}{ab} = 1 \Rightarrow \frac{k}{b} + \frac{h}{a} = 1$

⇒ $\frac{h}{a} + \frac{k}{b} = 1$

19. Find the equation of the straight line passing through the point $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$.

Sol: Let A = $(at_1^2, 2at_1)$ B = $(at_2^2, 2at_2)$

Slope of AB = $\frac{2at_2 - 2at_1}{at_2^2 - at_1^2} = \frac{2a(t_2 - t_1)}{a(t_2^2 - t_1^2)}$

= $\frac{2(t_2 - t_1)}{(t_2 - t_1)(t_2 + t_1)} = \frac{2}{t_2 + t_1}$

∴ equation of the straight line is $y - 2at_1 = \frac{2}{t_2 + t_1}(x - at_2^2)$

⇒ $y(t_2 + t_1) - 2at_1(t_2 + t_1) = 2x - 2at_1^2$

⇒ $2x - y(t_1 + t_2) + 2at_1t_2 = 0$

THE STRIGHT LINE (SAQ) (4 Marks)

1. Transform the following equations into (a) slope – intercept form (b) intercept form (c) normal form.

Sol: (i) $3x + 4y = 5$

Slope intercept form: $3x + 4y = 5$

$\Rightarrow 4y = 5 - 3x$

$\Rightarrow y = \frac{5}{4} - \frac{3}{4}x$

Slope (m) = $-\frac{3}{4}$ y-intercept - $\frac{5}{4}$

Intercept form: $3x + 4y = 5$

$\Rightarrow \frac{3x}{5} + \frac{4y}{5} = 1 \Rightarrow \frac{x}{\left(\frac{5}{3}\right)} + \frac{y}{\left(\frac{5}{4}\right)} = 1$

Normal form: $3x + 4y = 5$

Divide by $\sqrt{9+16} = 5$ on both sides

$\frac{3}{5}x + \frac{4}{5}y = 1$ which is of the form $x\ cos\alpha + y\ sin\alpha = P$

$cos\alpha = \frac{3}{5}$ and $sin\alpha = \frac{4}{5}$

(ii) $\sqrt{3}x + y = 4$

Sol: Slope – intersection form: $y - \sqrt{3}x + 4$

Slope = $-\sqrt{3}$ y-intercept = 4

Intercept form: $\sqrt{3}x + y = 4$

$\Rightarrow \frac{\sqrt{3}}{4}x + \frac{y}{4} = 1 \Rightarrow \frac{x}{\frac{4}{\sqrt{x}}} + \frac{y}{4} = 1$

Normal form: $\sqrt{3}x + y = 4$

Divide by $\sqrt{3+1} = 2$ on both sides

$\Rightarrow \frac{\sqrt{3}}{2}x + \frac{y}{2} = 2 \Rightarrow xcos\left(\frac{\pi}{6}\right) + ysin\left(\frac{\pi}{6}\right) = 2$

2. Transform the equation $\frac{x}{a}+\frac{y}{b}=1$ into the normal form when a > 0 and b > 0. If the perpendicular distance of the straight line from the origin is P deduce that $\frac{1}{P^2}=\frac{1}{a^2}+\frac{1}{b^2}$.

Sol: equation of the line is $\frac{x}{a}+\frac{y}{b}=1$

$$\Rightarrow bx+ay=ab$$

Divide on both sides with $\sqrt{a^2+b^2}$

$$\Rightarrow \frac{b}{\sqrt{a^2+b^2}}x+\frac{a}{\sqrt{a^2+b^2}}y=\frac{ab}{\sqrt{a^2+b^2}}$$

Which is of the form $x\,cos\alpha+y\,sin\alpha=P$

P is the perpendicular distance from the origin to the line $\frac{x}{a}+\frac{y}{b}=1 \Rightarrow \frac{x}{a}+\frac{y}{b}-1=0$

$$P=\frac{|-1|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}} \Rightarrow P=\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}} \Rightarrow \sqrt{\frac{1}{a^2}+\frac{1}{b^2}}=\frac{1}{P}$$

Squaring on both sides

$$\frac{1}{P^2}=\frac{1}{a^2}+\frac{1}{b^2}$$

3. If the product of the intercepts made by the straight line $x\,tan\alpha+y\,sec\alpha=1$ $\left(0<\alpha<\frac{\pi}{2}\right)$ on the Coordinate axes is equal to $\sin\alpha$, find α.

Sol: Give line is $x\,tan\alpha+y\,sec\alpha=1$

$$\Rightarrow \frac{x}{cot\alpha}+\frac{y}{cos\alpha}=1$$

Product of the intercept = $sin\alpha$

$$cot\alpha.cos\alpha=sin\alpha \Rightarrow \frac{cos\alpha}{sin\alpha},cos\alpha=sin\alpha$$

$$\Rightarrow cos^2\alpha=sin^2\alpha \Rightarrow \alpha=\frac{\pi}{4}.$$

4. Find the points on the line $3x-4y-1=0$ which are at a distance of 5 units from the point (3, 2).

Sol: equation of the line is symmetric form is

$$\frac{x-3}{cos\alpha}=\frac{y-2}{sin\alpha}=1$$

Coordinate of the point P are

$(3+r\,sin\alpha, 2+r\,sin\alpha)$ = $(3+5cos\alpha, 2+5\,sin\alpha)$

P is a point on $3x-4y-1=0$

$$3\,(3+5\,cos\alpha)-4(2+5\,sin\alpha)-1=0$$

$$\Rightarrow 9+15cos\alpha-8-20sin\alpha-1=0$$

$\Rightarrow 15cos\alpha = 20sin\alpha \Rightarrow tan\alpha = \frac{3}{4}.$

Case (i) $cos\alpha = \frac{4}{5} \quad sin\alpha = \frac{3}{5}$ then

Coordinating P are $\left[3 + 5\left(\frac{4}{5}\right) \ \ 2 + 5\left(\frac{3}{5}\right)\right] = (7, 5)$

Case (ii) $cos\alpha = \frac{-4}{5}$ $sin\alpha = \frac{-3}{5}$ then

Coordinating P are $\left[3 + 5\left(\frac{-4}{5}\right) \ \ 2 + 5\left(\frac{-3}{5}\right)\right] = (-1, -1)$

5. A straight line whose inclination with the positive direction of the x-axis measured in the anticlock wise sense is $\pi/3$ makes positive intercept on the y-axis. If the straight line is at a distance of 4 from the origin find its equation.

Sol: Given $\alpha = \frac{\pi}{3}$ P = 4

$m = tan\alpha = tan60^o = \sqrt{3}$

Equation of the line in the slope intercept form is

$y = \sqrt{3}\,x + c \ \Rightarrow \sqrt{3}x - y + c = 0$

Distance from the origin = 4

$\frac{|0-0+c|}{\sqrt{3+1}} = 4 \Rightarrow |c| = 8 \Rightarrow c = \pm 8$

Given c > 0 ∴ c = 8

∴ equation of the line is $\sqrt{3}x - y + 8 = 0$

6. Find the value of P. If the following lines are concurrent $3x + 4y = 5$, $2x + 3y = 4$, $Px + 4y = 6$,

Sol: given lines are $3x + 4y = 5$ ------- (1)

$2x + 3y = 4$ -------- (2)

$Px + 4y = 6$ -------- (3)

Intersecting point of (1) & (2)

	x		y		1	
4	⤫	-5	⤫	3	⤫	4
3		-4		2		3

$\Rightarrow \frac{x}{-16+15} = \frac{y}{-10+12} = \frac{1}{9-8} \Rightarrow \frac{x}{-1} = \frac{y}{2} = \frac{1}{1} \Rightarrow x = -1\,,\ y = 2$

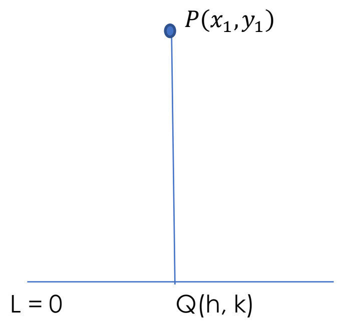
Since the lines are concurred, the point (-1, 2) lies on (3)

$\Rightarrow P(-1) + 4(2) - 6 = 0 \Rightarrow -P + 8 - 6 = 0$

$\Rightarrow -P + 2 = 0 \Rightarrow -P = -2 \ \Rightarrow P = 2$

7. If Q(h, k) is the foot of the perpendicular from $P(x_1, y_1)$ on the straight line $ax + by + c = 0$ then $(h - x_1): a = (k - y_1): b = -(ax_1 + by_1 + c) : \ a^2 + b^2$.

Sol: $L \equiv ax + by + c = 0$, $P(x_1, y_1) \quad Q(h, k)$



Slope of $L(m_1) = -\frac{a}{b}$

Slope of PQ(m2) = $\frac{k-y_1}{h-x_1}$

L ⊥ PQ ↔ m1m2 = -1

$\left(-\frac{a}{b}\right)\left(\frac{k-y_1}{h-x_1}\right) = -1$

$\frac{k-y_1}{h-x_1} = \frac{b}{a} \Rightarrow \frac{k-y_1}{b} = \frac{h-x_1}{a}$

$\frac{h-x_1}{a} = \frac{k-y_1}{b} = \lambda$ say ---------(1)

$h - x_1 = a\lambda \qquad k - y_1 = b\lambda$

$h = x_1 + a\lambda \qquad k = y_1 + b\lambda$

Q(h, k) is a point on L = 0

$\therefore a(x_1 + a\lambda) + b(y_1 + b\lambda) + c = 0$

$ax_1 + a^2\lambda + by_1 + b^2\lambda + c = 0$

$(a^2 + b^2)\lambda = -ax_1 - by_1 - c$

$\lambda = \frac{-(ax_1+by_1+c)}{a^2+b^2}$

∴ (1) becomes

$\frac{h-x_1}{a} = \frac{k-y_1}{b} = \frac{-(ax_1+by_1+c)}{a^2+b^2}$

8. Find the foot of the perpendicular from (-1, 3) on the straight line $5x - y - 18 = 0$.

Sol: Let Q(h, k) be the foot of the perpendicular from (-1, 3) to $5x - y - 18 = 0$

$\frac{h-x_1}{a} = \frac{k-y_1}{b} = \frac{-(ax_1+by_1+c)}{a^2+b^2}$

$\Rightarrow \frac{h-(-1)}{a} = \frac{k-3}{-1} = \frac{-(-5-3-18)}{5^2+1^2} \Rightarrow \frac{h+1}{5} = \frac{k-3}{-1} = \frac{26}{26} = 1$

$\Rightarrow \frac{h+1}{5} = 1 \quad \frac{k-3}{-1} = 1$

$\Rightarrow h = 5 - 1 \quad k = -1 + 3$

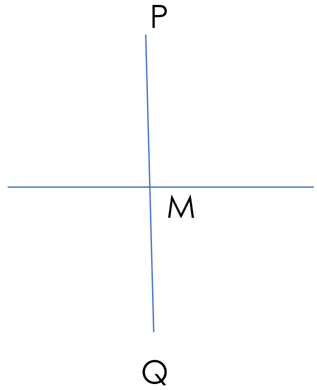
$\Rightarrow h = 4 \qquad k = 2$

$\therefore (h, k) = (4, 2)$

9. If Q (h, k) is the image of the point $P(x_1, y_1)$ with respect to the straight line $ax + by + c = 0$ then $(h - x_1): a = (k - y_1): b = -(ax_1 + by_1 + c) : a^2 + b^2$.

Sol: $L \equiv ax + by + c = 0$, $P(x_1, y_1)$ $Q(h, k)$

Slope of $L(m_1) = -\frac{a}{b}$



Slope of PQ(m_2) = $\frac{k-y_1}{h-x_1}$

L ⊥ PQ ↔ $m_1m_2 = -1$

$\left(-\frac{a}{b}\right)\left(\frac{k-y_1}{h-x_1}\right) = -1$

$\frac{k-y_1}{h-x_1} = \frac{b}{a} \Rightarrow \frac{k-y_1}{b} = \frac{h-x_1}{a}$

$\frac{h-x_1}{a} = \frac{k-y_1}{b} = \lambda$ say ---------(1)

$h - x_1 = a\lambda \qquad k - y_1 = b\lambda$

$h = x_1 + a\lambda \qquad k = y_1 + b\lambda$

M is the midpoint of PQ

$M = \left(\frac{x_1+h}{2}, \frac{y_1+k}{2}\right)$

It is a point on the line L = 0

$\therefore a\left(\frac{x_1+h}{2}\right) + b\left(\frac{y_1+k}{2}\right) + c = 0$

$ax_1 + a(x_1 + a\lambda) + by_1 + b(y_1 + b\lambda) + 2c = 0$

$ax_1 + ax_1 + a^2\lambda + by_1 + by_1 + b^2\lambda + 2c = 0$

$(a^2 + b^2)\lambda = -2ax_1 - 2by_1 - 2c$

$\lambda = \frac{-2(ax_1+by_1+c)}{a^2+b^2}$

$\therefore$ (1) becomes

$\frac{h-x_1}{a} = \frac{k-y_1}{b} = \frac{-2(ax_1+by_1+c)}{a^2+b^2}$

10. Find the image of (1, -2) w.r.to the straight line $2x - 3y + 5 = 0$.

Sol: (h, k) is the image of (1, -2) w.r. to the line $2x - 3y + 5 = 0$

$\therefore \frac{h-1}{2} = \frac{k+2}{-3} = \frac{-2[2(1)-3(-2)+5]}{(2)^2+(-3)^2}$

$\Rightarrow \frac{h-1}{2} = \frac{k+2}{-3} = \frac{-2[2+6+5]}{4+9}$

$\Rightarrow \frac{h-1}{2} = \frac{k+2}{-3} = \frac{-2[13]}{13}$

$\frac{h-1}{2} = -2 \Rightarrow h - 1 = -4 \Rightarrow h = -3$

$\frac{k+2}{-3} = -2 \Rightarrow k + 2 = 6 \Rightarrow k = 4$

$\therefore (h, k) = (-3, 4)$

11. A straight line through Q($\sqrt{3}, 2$) makes an angle $\frac{\pi}{6}$ with the positive direction of x-axis. If the straight line intersection the line $\sqrt{3}x - 4y + 8 = 0$ at P. find the distance PQ.

Sol: Given Q($\sqrt{3}, 2$) = (x_1, y_1)

Let $\theta = \frac{\pi}{6} = 30^o$ and PQ = r

$P(x_1 + r\,cos\theta,\ y_1 + r\,sin\theta) = \left(\sqrt{3} + r\cos 30^o, 2 + r\,sin 30^o\right)$

$= \left(\sqrt{3} + \frac{\sqrt{3}}{2}r,\ 2 + \frac{r}{2}\right)$

P lies on the line $\sqrt{3}x - 4y + 8 = 0$

$= \sqrt{3}\left(\sqrt{3} + \frac{\sqrt{3}}{2}r\right) - 42 + \frac{r}{2} + 8 = 0$

$\Rightarrow 3 + \frac{3}{2}r - 8 - \frac{4r}{2} + 8 = 0$

$\Rightarrow 3 - \frac{r}{2} = 0 \Rightarrow \frac{r}{2} = 3 \Rightarrow r = 6$

$\therefore PQ = 6$

12. Find the equation of straight line passing through the origin and also through the point of intersecting of the $2x - y + 5 = 0$, $x + y + 1 = 0$

Sol: Equation of the straight line passing through the point of intersecting L_1, L_2 is

$L_1 + \lambda L_2 = 0$

Equation of required line is

$2x - y + 5 + \lambda(x + y + 1) = 0$ ------------(1)

Passes through origin (0, 0)

$\Rightarrow (2(0) - 0 + 5) + \lambda(0+0+1) = 0$

$\Rightarrow 5 + \lambda = 0 \Rightarrow \lambda = -5$

$\therefore$ required line $2x - y + 5 - 5(x + y + 1) = 0$

$\Rightarrow 2x - y + 5x - 5y - 5 = 0$

$\Rightarrow -3x - 6y = 0 \Rightarrow x + 2y = 0$

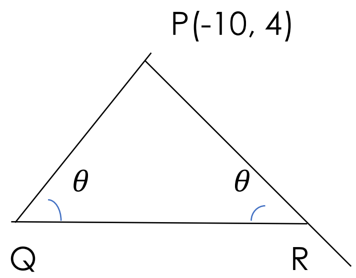
LAQ (7 Marks Questions)

1. Find the equation of the straight lines passing through the point (-10, 4) and making on angle 'θ' with the line $x - 2y = 10$ such that $tan\theta = 2$.

Sol: equation of QR is $x - 2y = 10$

Slope PQ = m, PQ passes through (–10, 4)

Equation of PQ is $y - 4 = m(x + 10)$



$\Rightarrow y - 4 = mx + 10m$

$\Rightarrow mx - y - 10m + 4 = 0$

Given that $tan\theta = 2$

$\therefore cos\theta = \frac{1}{\sqrt{5}}$

$cos\theta = \frac{|a_1a_2+b_1b_2|}{\sqrt{a_1^2+b_1^2}.\sqrt{a_2^2+b_2^2}} \Rightarrow \frac{1}{\sqrt{5}} = \frac{|m+2|}{\sqrt{1+4}.\sqrt{m^2+1}}$

$\Rightarrow \frac{1}{\sqrt{5}} = \frac{|m+2|}{\sqrt{5}\sqrt{m^2+1}} \Rightarrow \sqrt{m^2+1} = |m+2|$

Squaring on both sides we get

$m^2 + 1 = (m+2)^2 \Rightarrow m^2 + 1 = m^2 + 4m + 4$

$\Rightarrow 4m + 3 = 0 \Rightarrow m = \frac{-3}{4}$

Case 1: Coefficient of $m^2 = 0$

⇒ one of the root is ∞

Hence PR is a vertical line

Equation of PR is $x + 10 = 0$

Case 2: $m = \frac{-3}{4}$

Equation of PQ is

$y - 4 = \frac{-3}{4}(x + 10)$

$\Rightarrow 4y - 16 = -3x - 30 \Rightarrow 3x + 4y - 16 + 30 = 0$

$\Rightarrow 3x + 4y + 14 = 0$

2. The base of an equilateral triangle is $x + y - 2 = 0$ and the opposite vertex is (3, -1). Find the equation of the remaining sides.

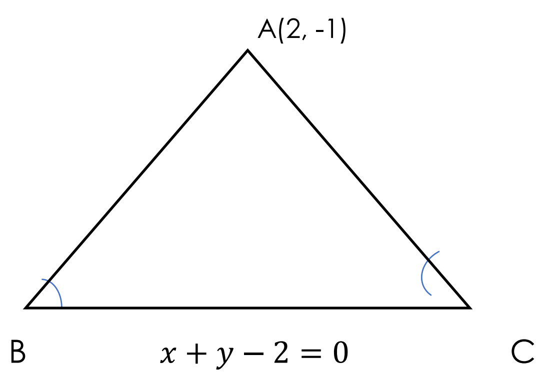
Sol: equation of BC is $x + y - 2 = 0$

AB passes through A(2, –1)

Slope of AB = m

Equation of AB is $y + 1 = m(x - 2)$

$\Rightarrow mx - y - (2m + 1) = 0$



$cos60^o = \frac{|m-1|}{\sqrt{1+1}.\sqrt{m^2+1}} \Rightarrow \frac{1}{\sqrt{2}} = \frac{|m-1|}{\sqrt{2}.\sqrt{m^2+1}}$

Squaring and cross multiplying

$2(m^2 + 1) = 4(m - 1)^2 \Rightarrow m^2 + 1 = 2(m^2 - 2m + 1)$

$\Rightarrow m^2 + 1 = 2m^2 - 4m + 2 \Rightarrow m^2 - 4m + 1 = 0$

$m = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$

$\Rightarrow m = 2 \pm \sqrt{3}$

equation of AB $\Rightarrow y + 1 = (2 + \sqrt{3})(x - 2)$

equation of AC $\Rightarrow y + 1 = (2 + \sqrt{3})(x - 2)$.

3. Find the equation of the straight lines passing through the point (-3, 2) and making an angle 45° with the straight line $3x - y + 4 = 0$.

Sol: Given point P(–3, 2)

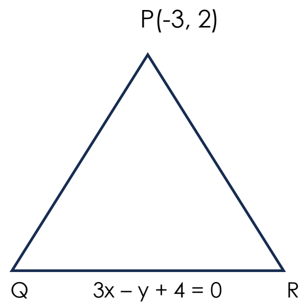
Given line is $3x - y + 4 = 0$ --------- (1)

Let M be the slope of a line which passes through P

Equation of line passing through P(–3, 2) having slope m is

$y - 2 = m(x + 3) \Rightarrow mx - y + 3m + 2 = 0$ ---------- (2)

Angle between (1) & (2)



$cos 45^o = \frac{|3m+1|}{\sqrt{9+1}.\sqrt{m^2+1}} \Rightarrow \frac{1}{\sqrt{2}} = \frac{|3m+1|}{\sqrt{10}.\sqrt{m^2+1}}$

Squaring on both sides

$\frac{1}{2} = \frac{9m^2+6m+1}{10(m^2+1)} \Rightarrow 10(m^2 + 1) = 18m^2 + 12m + 2$

$\Rightarrow 10m^2 + 10 = 18m^2 + 12m + 2$

$\Rightarrow 8m^2 + 12m - 8 = 0$

$\Rightarrow 2m^2 + 3m - 2 = 0$

$\Rightarrow (m + 2)(2m - 1) = 0$

$m = -2$ $\qquad$ $m = \frac{1}{2}$

Case1: $m = -2$ then equation (2) becomes

$y - 2 = -2(x + 3) \Rightarrow 2x + y + 4 = 0$

Case 2: $m = \frac{1}{2}$ then

$y - 2 = \frac{1}{2}(x + 3) \Rightarrow 2y - 4 = x + 3$

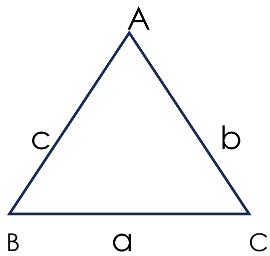
$\Rightarrow x - 2y + 7 = 0$

4. Find the incenter of the triangle whose vertices are (1, $\sqrt{3}$), (2, 0) and (0, 0).

Sol: A(0, 0) B(2, 0) c(1, $\sqrt{3}$)

c = AB = $\sqrt{(2-0)^2 + (0-0)^2} = \sqrt{4} = 2$

b = = AC = $\sqrt{(1-0)^2 + \left(\sqrt{3} - 0\right)^2} = \sqrt{1+3} = 2$



a = BC = $\sqrt{(1-2)^2 + \left(\sqrt{3} - 0\right)^2} = \sqrt{1+3} = 2$

intersect $I = \left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$

= $\left(\frac{2(0)+2(2)+2(1)}{2+2+2}, \frac{2(0)+2(0)+2\sqrt{3}}{2+2+2}\right) = \left(\frac{4+2}{6}, \frac{2\sqrt{3}}{6}\right)$

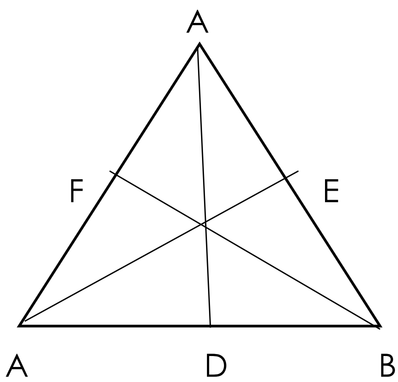
= $\left(\frac{6}{6}, \frac{2\sqrt{3}}{6}\right)$ = $\left(1, \frac{1}{\sqrt{3}}\right)$

$\therefore I = \left(1, \frac{1}{\sqrt{3}}\right)$

5. Find the orthocenter of the triangle whose vertices are (-5, -7), (13, 2) and (-5, 6).

Sol: A(−5, −7) B(13, 2) C(−5,6)

Slope of BC = $\frac{6-2}{-5-13} = \frac{4}{-18} = -\frac{2}{9}$



AD ⊥ BC

Slope of AD = 9/2

Equation of AD $\Rightarrow y + 7 = \frac{9}{2}(x + 5)$

$\Rightarrow 2y + 14 = 9x + 45$

$\Rightarrow 9x - 2y + 31 = 0$ --------- (1)

Slope of AC = $\frac{6+7}{-5+5} = \frac{13}{0} = \infty$

AC ⊥ BE

Slope of BE = 0

Equation of BE ⇒ $y - 2 = 0(x - 13)$

⇒ $y - 2 = 0$

⇒ $y - 2 = 0$ --------- (2)

Intersecting point of (1) & (2) is our required orthocenter from (1) & (2)

$9x - 2(2) + 31 = 0$

⇒ $9x + 27 = 0$ ⇒ $9x = -27$

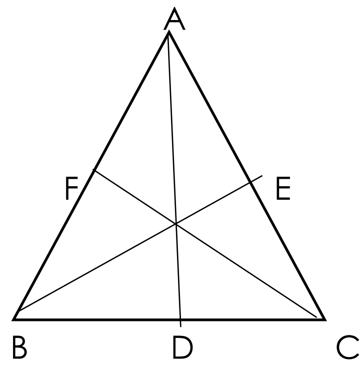
⇒ $x = -3$

∴ orthocenter (o) = (–3, 2)

6. Find the orthocenter of the triangle whose vertices are (5, -2), (-1, 2), (1, 4).

Sol: A$(5, -2)$ B(–1, 2) C$(1,4)$

Let O be the orthocenter



Slope of BC = $m = \frac{4-2}{1+1} = \frac{2}{2} = 1$

Slope of AD = –1

Equation of AD ⇒ $y + 2 = -1(x - 5)$

⇒ $y + 2 = -x + 5$

⇒ $x + y - 3 = 0$ --------- (1)

Slope of AC = $\frac{4+2}{1-5} = \frac{6}{-4} = -\frac{3}{2}$

Slope of BE = $\frac{2}{3}$

Equation of BE ⇒ $y - 2 = \frac{2}{3}(x + 1)$

⇒ $3y - 6 = 2x + 2$

⇒ $2x - 3y + 8 = 0$ --------- (2)

Solving (1) & (2) we get

x y 1

1   -3   1   1

-3   8   2   -3

$\Rightarrow \frac{x}{8-9} = \frac{y}{-6-8} = \frac{1}{-3-2} \Rightarrow \frac{x}{-1} = \frac{y}{-14} = \frac{1}{-5}$

$\frac{x}{-1} = \frac{1}{-5} \Rightarrow x = \frac{1}{5}$, $\quad \frac{y}{-14} = \frac{1}{-5} \Rightarrow y = \frac{14}{5}$

∴ orthocenter (O) = $\left(\frac{1}{5}, \frac{14}{5}\right)$

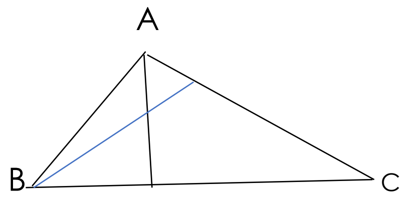
7. If the equation s of the sides of a triangle are $7x + y - 10 = 0$, $x - 2y + 5 = 0$ and $x + y + 2 = 0$, find the orthocenter of the triangle.

Sol: Given equations

BC : $7x + y - 10 = 0$ -------- (1)

AC : $x - 2y + 5 = 0$ --------- (2)

AB : $x + y + 2 = 0$ ------------(3)



for 'A' solving (2) & (3)

(2) – (3) ⇒ $x - 2y + 5 - (x + y + 2) = 0 \Rightarrow x - 2y + 5 - x - y - 2 = 0$

$\Rightarrow -3y + 3 = 0 \Rightarrow y = 1$

$y = 1$ in (2) ⇒ $x - 2(1) + 5 = 0 \Rightarrow x - 2 + 5 = 0 \Rightarrow x = -3$

∴ A = (–3, 1)

for 'B' solving (1) & (3)

(1) – (3) ⇒ $7x + y - 10 - (x + y + 2) = 0 \Rightarrow 7x + y - 10 - x - y - 2 = 0$

$\Rightarrow 6x - 12 = 0 \Rightarrow x = 2$

$x = 2$ in (3) ⇒ $2 + y + 2 = 0 \Rightarrow y = -4$

∴ B = (2, –4)

Slope of BC = –7

BC ⊥ AD ⇒ Slope of AD = $\frac{1}{7}$

equation of AD $\Rightarrow y-1=\frac{1}{7}(x+3)$

$7y-7=x+3 \Rightarrow x-7y+10=0$ ---------- (4)

Slope of AC = $\frac{1}{2}$ AC ⊥ BE

Slope of BE = –2

equation of BE $\Rightarrow y+4=-2(x-2) \Rightarrow y+4=-2x+4$

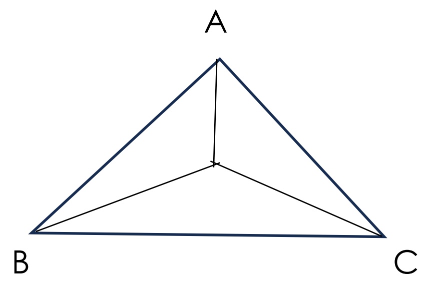
$\Rightarrow 2x+y=0$ ----------- (5)

Solving (4) & (5) we get

Orthocenter (O) = $\left(-\frac{2}{3},\frac{4}{3}\right)$

8. Find the circumcenter of the triangle whose vertices are (1, 3), (0, -2) and (-3, 1).

Sol: Let A(1, 3) B(0, –2) C(–3, 1)



Let $S(x,y)$ be the circumcenter

We know that SA = SB = SC

SA = SB

$\sqrt{(x-1)^2+(y-3)^2}=\sqrt{(x-0)^2+(y+2)^2}$

Squaring on both sides

$\Rightarrow x^2+1-2x+y^2-6y+9=x^2+y^2+4y+4$

$\Rightarrow -2x-6y+10=4y+4$

$\Rightarrow 2x+10y-6=0$

$\Rightarrow x+5y-3=0$ ------- (1)

SB = SC

$\Rightarrow \sqrt{(x-0)^2+(y+2)^2}=\sqrt{(x+3)^2+(y-1)^2}$

Squaring on both sides

$\Rightarrow x^2+y^2+4y+4=x^2+6x+9+y^2-2y+1$

$\Rightarrow 4y+4=6x-2y+10$

$\Rightarrow 6x - 6y + 6 = 0$

$\Rightarrow x - y + 1 = 0$ ------- (2)

Solving (1) & (2)

eq (1) – eq (2) $\Rightarrow x + 5y - 3 - (x - y + 1) = 0 \Rightarrow 6y - 4 = 0$

$\Rightarrow 6y = 4 \Rightarrow y = \frac{2}{3}$

Put $y = \frac{2}{3}$ in $x - y + 1 = 0 \Rightarrow x - \frac{2}{3} + 1 = 0 \Rightarrow x + \frac{1}{3} = 0$

$\Rightarrow x = \frac{-1}{3}$

∴ Circumcenter $S(x, y) = \left(\frac{-1}{3}, \frac{2}{3}\right)$

9. Find the circumcenter of the triangle whose vertices are (1, 3), (-3, 5) and (5, -1).

Sol: Let A(1, 3) B(–3, 5) C(5, –1)

Let $S(x, y)$ be the circumcenter

We know that SA = SB = SC

SA = SB

$\sqrt{(x-1)^2 + (y-3)^2} = \sqrt{(x+3)^2 + (y-5)^2}$

Squaring on both sides

$\Rightarrow x^2 + 1 - 2x + y^2 - 6y + 9 = x^2 + 6x + 9 + y^2 - 10y + 25$

$\Rightarrow 8x - 4y + 24 = 0$

$\Rightarrow 2x - y + 6 = 0$ ------- (1)

SB = SC

$\Rightarrow \sqrt{(x+3)^2 + (y-5)^2} = \sqrt{(x-5)^2 + (y+1)^2}$

Squaring on both sides

$\Rightarrow x^2 + +6x + 9 + y^2 - 10y + 25 = x^2 - 10x + 25 + y^2 + 2y + 1$

$\Rightarrow 16x - 12y + 8 = 0$

$\Rightarrow 4x - 3y + 2 = 0$ ------- (2)

Solving (1)& (2)

	x	y	1
-1	6	2	-1
-3	2	4	-3

$\Rightarrow \frac{x}{-2+18} = \frac{y}{24-4} = \frac{1}{-6+4} \Rightarrow \frac{x}{16} = \frac{y}{20} = \frac{1}{-2}$

$\Rightarrow x = \frac{16}{-2} = -8, y = \frac{20}{-2} = -10$

$\therefore$ Circumcenter $S(x, y) = (-8, -10)$

10. If p and q are the length of the perpendiculars from the origin to the straight lines $x\ sec\alpha + y\ cosec\ \alpha = a$ and $x\ cos\alpha - ysin\alpha = a\ cos2\alpha$. Prove that $4p^2 + q^2 = a^2$.

Sol: Given straight lines

$x\ sec\alpha + y\ cosec\ \alpha = a \Rightarrow x\ sec\alpha + y\ cosec\ \alpha - a = 0$ -------- (1)

$x\ cos\alpha - ysin\alpha = a\ cos2\alpha \Rightarrow x\ cos\alpha - ysin\alpha - a\ cos2\alpha = 0$ ------- (2)

p = Length of perpendicular distance from (0, 0) to (1) is

$\Rightarrow \text{p} = \frac{|0(sec\alpha + 0\ (cosec\alpha) - a)|}{\sqrt{sec^2\alpha + cosec^2\alpha}} = \frac{|-a|}{\sqrt{\frac{1}{cos^2\alpha} + \frac{1}{sin^2\alpha}}}$

$\Rightarrow \frac{a}{\sqrt{\frac{sin^2\alpha + cos^2\alpha}{sin^2\alpha . cos^2\alpha}}} = \frac{asin\alpha . cos\alpha}{\sqrt{sin^2\alpha + cos^2\alpha}} \Rightarrow p = a\ sin\alpha . cos\alpha$

q = Length of perpendicular distance from (0, 0) to the line (2)

$\Rightarrow q = \frac{|0 - 0 - a\ cos2\alpha|}{\sqrt{cos^2\alpha + sin^2\alpha}} = \frac{a\ cos2\alpha}{\sqrt{1}}$

$\Rightarrow q = a\ cos2\alpha$

$4p^2 + q^2 = 4(a\ sin\alpha . cos\alpha)^2 + (a\ cos2\alpha)^2$

$= 4a^2cos^2\alpha sin^2\alpha + a^2cos^2 2\alpha \Rightarrow a^2[(2\ sin\alpha\ cos\alpha)^2 + cos^2 2\alpha]$

$= a^2[sin^2 2\alpha + cos^2 2\alpha] = a^2(1) = a^2$

$\therefore 4p^2 + q^2 = a^2$

4. PAIR OF STRAIGHT LINES

(Weightage: 7 + 7 = 14M)

KEY POINTS

- If $h^2 \geq ab$ then $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines passing through the origin.
- Let $ax^2 + 2hxy + by^2 = (l_1x + m_1y)(l_2x + m_2y)$

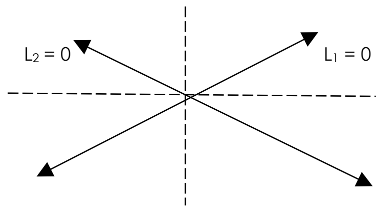
$$= l_1l_2x^2 + (l_1m_2 + l_2m_1)xy + m_1m_2y^2 = 0$$

Comparing on both sides ,

$$a = l_1l_2,\ 2h = l_1m_2 + l_2m_1,\ b = m_1m_2$$

- The equation of bisects of angles between the lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ are

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$



- The equation of bisects of angles between the lines $ax^2 + 2hxy + by^2 = 0$ is $h(x^2 - y^2) = (a - b)xy$.
- If θ is the acute angle between the pair of lines $ax^2 + 2hxy + by^2 = 0$ then,

(i) $cos\theta = \frac{|a+b|}{\sqrt{(a-b)^2+4h^2}}$ ii) $tan\theta = \frac{2\sqrt{h^2-ab}}{|a+b|}$

(iii) The above lines are perpendicular $\Leftrightarrow a \neq b = 0$

(iv) The above lines are coincide $\Leftrightarrow h^2 - ab = 0$

- The equation of the lines passing through (x_1, y_1) and

(i) Parallel to $ax^2 + 2hxy + by^2 = 0$ is $a(x - x_1)^2 + 2h(x - x_1)(y - y_1) + b(y - y_1)^2 = 0$

(ii) Perpendicular to $ax^2 + 2hxy + by^2 = 0$ is $b(x - x_1)^2 - 2h(x - x_1)(y - y_1) + a(y - y_1)^2 = 0$

- The area of the triangle formed by $ax^2 + 2hxy + by^2 = 0$ and the line $lx + my + n = 0$ is $\frac{n^2\sqrt{h^2-ab}}{|am^2-2hlm+bl^2|}$
- The product of perpendicular drawn from a point to the pair of lines $ax^2 + 2hxy + by^2 = 0$ is $\left|\frac{a\alpha^2+2h\alpha\beta+b\beta^2}{\sqrt{(a-b)^2+4h^2}}\right|$.
- If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two pairs of lines then

(i) Δ= $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

(ii) $h^2 \geq ab,\ g^2 \geq ac,\ f^2 \geq bc$

- The point of intersection of the lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is $\left(\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2}\right)$
- If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of parallel line then (i) $h^2 = ab$ (ii) $af^2 = bg^2$ and (iii) distance between these parallel lines is $2\sqrt{\frac{g^2-ac}{a(a+b)}} = 2\sqrt{\frac{f^2-ac}{b(a+b)}}$
- Homogenisation: The equation of the pair of lines joining the origin to the points of intersection of the curve $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and the line $lx + my + n = 0$ is

$$ax^2 + 2hxy + by^2 + 2gx\left(\frac{lx+my}{-n}\right) + 2fy\left(\frac{lx+my}{-n}\right) + c\left(\frac{lx+my}{-n}\right)^2 = 0$$

- The pair of lines of $ax^2 + 2hxy + by^2 = 0$ are at right angles,
Coefficient of x^2 + coefficient of y^2 = 0 (or) $a + b = 0$

LEVEL – I (7 Marks)

1. Theorem 1: Show that the equation to the pair of bisectors of angles between the pair of lines $ax^2 + 2hxy + by^2 = 0$ is $h(x^2 - y^2) = (a - b)xy$.

Sol: Let the equation $ax^2 + 2hxy + by^2 = 0$ by presents the lines

$$l_1x + m_1y = 0,\ l_2x + m_2y = 0$$

Then, $ax^2 + 2hxy + by^2 = (l_1x + m_1y)(l_2x + m_2y)$

$$= l_1l_2x^2 + (l_1m_2 + l_2m_1)xy + m_1m_2y^2$$

Comparing on both sides we get,

$$a = l_1l_2,\ 2h = l_1m_2 + l_2m_1,\ b = m_1m_2$$

Equation of the angular bisectors is $\frac{l_1x+m_1y}{\sqrt{l_1^2+m_1^2}} = \pm\frac{l_2x+m_2y}{\sqrt{l_2^2+m_2^2}}$

$$(l_1x + m_1y)\sqrt{l_2^2 + m_2^3} = \pm(l_2x + m_2y)\sqrt{l_1^2 + m_1^3}$$

Squaring on both sides,

$$(l_1x + m_1y)^2(l_2^2 + m_2^3) = \pm(l_2x + m_2y)^2(l_1^2 + m_1^3)$$

$(l_1^2x^2+m_1^2y^2+2l_1m_1xy)(l_2^2+m_2^3)=(l_2^2x^2+m_2^2y^2+2l_2m_2xy)(l_1^2+m_1^3)$

$l_1^2l_2^2x^2+l_2^2m_1^2y^2+2l_1l_2^2m_1xy+l_1^2m_2^2x^2+m_1^2m_2^2y^2+2l_1m_1m_2^2xy$

$=l_2^2l_1^2x^2+l_1^2m_2^2y^2+2l_1^2l_2m_2xy+l_2^2m_1^2x^2+m_1^2m_2^2y^2+2l_2m_1^2m_2xy$

$(l_1^2m_2^2-l_2^2m_1^2)(x^2-y^2)=2xy[l_1l_2(l_1m_2-l_2m_1)-m_1m_2(l_1m_2-l_2m_1)]$

$(l_1m_2+l_2m_1)(l_1m_2-l_2m_1)(x^2-y^2)=2xy[(l_1l_2-m_1m_2)(l_1m_2-l_2m_1)]$

$2h(x^2-y^2)=2xy(a-b)$

$\Rightarrow h(x^2-y^2)=xy(a-b)$

2. Theorem 2: Show that the product of perpendiculars from (α,β) to the pair of lines $ax^2+2hxy+by^2=0$ is $\left|\frac{a\alpha^2+2h\alpha\beta+b\beta^2}{\sqrt{(a-b)^2+4h^2}}\right|$,

Sol: Let the equation $ax^2+2hxy+by^2=0$ represents the lines

$l_1x+m_1y=0$ ------- (1)

$l_2x+m_2y=0$ -------- (2)

Then, $ax^2+2hxy+by^2=(l_1x+m_1y)(l_2x+m_2y)$

$=l_1l_2x^2+(l_1m_2+l_2m_1)xy+m_1m_2y^2$

Comparing on both sides we get

$a=l_1l_2,\ 2h=l_1m_2+l_2m_1,\ b=m_1m_2$

The length of perpendicular from (α,β) to line (1) is $\left|\frac{l_1\alpha+m_1\beta}{\sqrt{l_1^2+m_1^2}}\right|$

The length of perpendicular from (α,β) line (2) is $\left|\frac{l_2\alpha+m_2\beta}{\sqrt{l_2^2+m_2^2}}\right|$

∴ The product of perpendiculars = $\left|\frac{l_1\alpha+m_1\beta}{\sqrt{l_1^2+m_1^2}}\right|.\left|\frac{l_2\alpha+m_2\beta}{\sqrt{l_2^2+m_2^2}}\right|$

$=\left|\frac{l_1l_2\alpha^2+l_1m_2\alpha\beta+l_2m_1\alpha\beta+m_1m_2\beta^2}{\sqrt{l_1^2l_2^2+l_1^2m_2^2+l_2^2m_1^2+m_1^2m_2^2}}\right|$

$$= \left|\frac{l_1l_2\alpha^2+(l_1m_2+l_2m_1)\alpha\beta+m_1m_2\beta^2}{\sqrt{(l_1l_2-m_1m_2)^2+2l_1l_2m_1m_2+(l_1l_2+m_1m_2)^2-2l_1l_2m_1m_2}}\right|$$

$$= \left|\frac{a\alpha^2+2h\alpha\beta+b\beta^2}{\sqrt{(a-b)^2+4h^2}}\right|$$

3. Theorem 3: If θ is the angle between the pair of lines represented by $ax^2 + 2hxy + by^2 = 0$ then prove that $cos\theta = \left|\frac{a+b}{\sqrt{(a-b)^2+4h^2}}\right|$.

Sol: Let the equation $ax^2 + 2hxy + by^2 = 0$ represents the lines $l_1x + m_1y = 0$, $l_2x + m_2y = 0$

$$x^2 + 2hxy + by^2 = (l_1x + m_1y)(l_2x + m_2y)$$

$$= l_1l_2x^2 + (l_1m_2 + l_2m_1)xy + m_1m_2y^2$$

Comparing on both sides we get

$$a = l_1l_2, \; 2h = l_1m_2 + l_2m_1, \; b = m_1m_2$$

If θ is the angle between the pair of lines then,

$$cos\theta = \frac{|l_1l_2+m_1m_2|}{\sqrt{l_1^2+m_1^2}\sqrt{l_2^2+m_2^2}} = \frac{|l_1l_2+m_1m_2|}{\sqrt{l_1^2l_2^2+l_1^2m_2^2+l_2^2m_1^2+m_1^2m_2^2}}$$

$$= \frac{|l_1l_2+m_1m_2|}{\sqrt{(l_1l_2-m_1m_2)^2+2l_1l_2m_1m_2+(l_1l_2+m_1m_2)^2-2l_1l_2m_1m_2}}$$

$$\therefore cos\theta = \frac{|a+b|}{\sqrt{(a-b)^2+(2h)^2}} = \left|\frac{a+b}{\sqrt{(a-b)^2+4h^2}}\right|$$

4. Theorem 4: Show that the area of the triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my + n = 0$ is $\frac{n^2\sqrt{h^2-ab}}{|am^2-2hlm+bl^2|}$.

Sol: Let the equation $ax^2 + 2hxy + by^2 = 0$ represents the lines

$l_1x + m_1y = 0$ ------------ (1)

$l_2x + m_2y = 0$ ----------- (2)

$$x^2 + 2hxy + by^2 = (l_1x + m_1y)(l_2x + m_2y)$$

$$= l_1l_2x^2 + (l_1m_2 + l_2m_1)xy + m_1m_2y^2$$

Comparing on both sides we get

$$a = l_1l_2, \; 2h = l_1m_2 + l_2m_1, \; b = m_1m_2$$

Let the given line $lx + my + n = 0$ --------- (3)

Solving eq (1) & (2) we get O(0, 0)

Solving eq (1) & (3) we get A:

	x	y	1	$\frac{x}{m_1n-0} = \frac{y}{0-nl_1} = \frac{1}{l_1m-lm_1}$
m_1	0	l_1	m_1	$\Rightarrow x = \frac{m_1n}{l_1m-lm_1}, y = \frac{-nl_1}{l_1m-lm_1}$
m	n	l	m	$\therefore A = \left(\frac{m_1n}{l_1m-lm_1}, \frac{-nl_1}{l_1m-lm_1}\right) = (x_1, y_1)$

Solving eq (2) & (3) we get B:

	x	y	1	$\frac{x}{m_2n-0} = \frac{y}{0-nl_2} = \frac{1}{l_2m-lm_2}$
m_2	0	l_2	m_2	$\Rightarrow x = \frac{m_2n}{l_2m-lm_2}, y = \frac{-nl_2}{l_2m-lm_2}$
m	n	l	m	$\therefore$ B$= \left(\frac{m_2n}{l_2m-lm_2}, \frac{-nl_2}{l_2m-lm_2}\right) = (x_2, y_2)$

$\therefore$ Area of ΔOAB with vertices O(0, 0), $A(x_1, y_1)$, $B(x_2, y_2)$ is $\Delta = \frac{1}{2}|x_1y_2 - x_2y_1|$

$$= \frac{1}{2}\left|\frac{m_1n}{l_1m-lm_1}.\frac{-nl_2}{l_2m-lm_2} - \frac{m_2n}{l_2m-lm_2}.\frac{-nl_1}{l_1m-lm_1}\right|$$

$$= \frac{1}{2}\left|\frac{l_1m_2n^2-l_2m_1n^2}{(l_1m-lm_1)(l_2m-lm_2)}\right| = \frac{1}{2}\left|\frac{n^2(l_1m_2-l_2m_1)}{l_1l_2m^2+lm(l_1m_2+l_2m_1)+m_1m_2n^2}\right|$$

$$= \frac{1}{2}\left|\frac{n^2\sqrt{(2h)^2-4ab}}{am^2-2hlm+bl^2}\right| = \frac{1}{2}\left|\frac{2\,n^2\sqrt{h^2-4ab}}{am^2-2hlm+bl^2}\right| = \left|\frac{n^2\sqrt{h^2-4ab}}{am^2-2hlm+bl^2}\right|$$

Area of ΔOAB = $\left|\frac{n^2\sqrt{h^2-4ab}}{am^2-2hlm+bl^2}\right|$

5. Theorem 5: If the Second degree equation $S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ in the two variables x and y represents a pair of straight lines then now that (i) $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ (ii) $h^2 \geq ab,\ g^2 \geq ac,\ f^2 \geq bc$

Sol: Let $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents pair of straight lines.

$l_1x + m_1y + n_1 = 0$ -------- (1)

$l_2x + m_2y + n_2 = 0$ ---------- (2)

$\therefore ax^2 + 2hxy + by^2 + 2gx + 2fy + c = (l_1x + m_1y + n_1)(l_2x + m_2y + n_2) = 0$

$= l_1l_2x^2 + (l_1m_2 + l_2m_1)xy + m_1m_2y^2 + (l_1n_2 + l_2n_1)x + (m_1n_2 + m_2n_1)y + n_1n_2 = 0$

Comparing on both sides

$a = l_1l_2,\ 2h = l_1m_2 + l_2m_1,\ 2g = l_1n_2 + l_2n_1,\ 2f = m_1n_2 + m_2n_1,\ c = n_1n_2$

Consider,

$(2f)(2g)(2h) = (m_1n_2 + m_2n_1)(l_1n_2 + l_2n_1)(l_1m_2 + l_2m_1)$

$8fgh = l_1l_2(m_1^2n_2^2 + m_2^2n_1^2) + m_1m_2(l_1^2n_2^2 + l_2^2n_1^2) + n_1n_2(l_1^2m_2^2 + l_2^2m_1^2) + 2l_1l_2m_1m_2n_1n_2$

$= l_1l_2[(m_1^2n_2^2 + m_2^2n_1^2)^2 - 2m_1m_2n_1n_2] + m_1m_2[(l_1^2n_2^2 + l_2^2n_1^2)^2 - 2l_1l_2n_1n_2] + n_1n_2[(l_1^2m_2^2 + l_2^2m_1^2)^2 - 2l_1l_2m_1m_2n_1n_2] + 2l_1l_2m_1m_2n_1n_2$

$= a[(2f)^2 - 2bc] + b[(2g)^2 - 2ac] + c[(2h)^2 - 2ab] + 2abc$

$8fgh = 4(af^2 + bg^2 + ch^2 - abc)$

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

(ii) $4h^2 - 4ab = (2h)^2 - 4ab = (l_1m_2 + l_2m_1)^2 - 4l_1l_2m_1m_2$

$= (l_1m_2 - l_2m_1)^2 \geq 0$

$\therefore 4h^2 \geq 4ab \Rightarrow h^2 \geq ab$

6. Theorem 6: If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of parallel lines then prove that (i) $h^2 = ab$ (ii) $af^2 = bg^2$ and (iii) the distance between the parallel lines $2\sqrt{\frac{g^2-ac}{a(a+b)}} = 2\sqrt{\frac{f^2-ac}{b(a+b)}}$

Sol: Let $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of parallel lines

$l_1x + m_1y + n_1 = 0$ -------- (1)

$l_2x + m_2y + n_2 = 0$ ---------- (2)

$\therefore ax^2 + 2hxy + by^2 + 2gx + 2fy + c = (lx + my + n_1)(lx + my + n_2) = 0$

$= l^2x^2 + 2lmxy + m^2y^2 + l(n_1 + n_2)x + m(n_1 + n_2)y + n_1n_2$

Comparing on both sides

$a = l^2,\ 2h = 2lm,\ b = m^2\ \ 2g = l(n_1 + n_2),\ 2f = m(n_1 + n_2),\ c = n_1n_2$

$\Rightarrow h = lm,\ g = \frac{l(n_1+n_2)}{2},\ f = \frac{m(n_1+n_2)}{2}$

(i) $h^2 = (lm)^2 = l^2m^2$

$ab = l^2m^2 \quad \therefore h^2 = ab$

(ii) $af^2 = l^2\left(\frac{m(n_1+n_2)}{2}\right)^2 = \frac{l^2m^2(n_1+n_2)^2}{4}$

$= m^2\left(\frac{l^2(n_1+n_2)^2}{4}\right) = bg^2$

$\therefore\ af^2 = bg^2$

(iii) The distance between the parallel line is $\frac{|c_1-c_2|}{\sqrt{l^2+m^2}}$

$= \frac{|n_1-n_2|}{\sqrt{l^2+m^2}} = \frac{\sqrt{\left((n_1+n_2)\right)^2-4n_1n_2}}{\sqrt{a+b}}$

$= \frac{\sqrt{\left(\frac{2g}{l}\right)^2-4c}}{\sqrt{a+b}} = \frac{\sqrt{\frac{4g^2}{l^2}-4c}}{\sqrt{a+b}}$

$= \frac{\sqrt{\frac{4g^2}{a}-4c}}{\sqrt{a+b}} = 2\sqrt{\frac{g^2-ac}{a(a+b)}}$

Similarly we can prove that,

Distance between the parallel lines = $2\sqrt{\frac{f^2-ac}{b(a+b)}}$

7. Show that the two pair of lines $6x^2-5xy-6y^2=0$ and $6x^2-5xy-6y^2+x+5y-1=0$ forms a square.

Sol: $6x^2-5xy-6y^2 = 6x^2-9xy+4xy-6y^2$

$= (3x+2y)(2x-3y)$

Let $6x^2-5xy-6y^2+x+5y-1 = (3x+2y+l)(2x-3y+m)$

$= 6x^2-5xy-6y^2+(3m+2l)x+(2m-3l)y+lm$

$3m+2l = 1 \Rightarrow 3m+2l-1 = 0$

$2m-3l = 5 \Rightarrow 2m-3l-5 = 0$

	m	l	1
2	-1	3	2
-3	-5	2	-3

$\frac{m}{-10-3} = \frac{l}{-2+15} = \frac{1}{-9-4}$

$\frac{m}{-13} = \frac{l}{13} = \frac{1}{-13}$

$m = 1, l = -1$

$\therefore 6x^2 - 5xy - 6y^2 + x + 5y - 1 = (3x + 2y - 1)(2x - 3y + 1)$

The above lines are mutually perpendicular and hence they form a rectangle.

The distance between pair of opposite lines $3x + 2y = 0$, $3x + 2y - 1 = 0$ is $\frac{1}{\sqrt{9+4}} = \frac{1}{\sqrt{13}}$

The distance between $2x - 3y = 0$ and $2x - 3y + 1 = 0$ is $\frac{1}{\sqrt{9+4}} = \frac{1}{\sqrt{13}}$

$\therefore$ the given equation form a square.

8. Prove that the equation $2x^2 - 13xy - 7y^2 + x + 23y - 6 = 0$ represents pair of straight lines. Also find the angle between them and the coordinates of the point of intersection of the lines.

Sol: Given pair of lines $2x^2 - 13xy - 7y^2 + x + 23y - 6 = 0$

Compare with a$x^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ then we get

$a = 2,\ b = -7,\ c = -6,\ h = \frac{-13}{2},\ g = \frac{1}{2},\ f = \frac{23}{2}$

If $\Delta = 0$ then the above equation represents two pair of lines.

Now, $\Delta = abc + 2fgh - af^2 - bf^2 - ch^2$

$= 2(-7)(-6) + 2\left[\frac{23}{2}\right]\left[\frac{1}{2}\right]\left[\frac{-13}{2}\right] - 2\left[\frac{23}{2}\right]^2 + 7\left[\frac{1}{2}\right]^2 + 6\left[\frac{-13}{2}\right]^2 = 0$

Also, $h^2 - ab = \left[\frac{-13}{2}\right]^2 - 2(-7) = \frac{169}{4} + 14 = \frac{169+56}{4} = \frac{225}{4} > 0 \Rightarrow h^2 > ab$

and $g^2 - ac = \left[\frac{1}{2}\right]^3 - 2(-6) = \frac{1}{4} + 12 = \frac{1+48}{4} = \frac{49}{4} > 0 \Rightarrow g^2 > ac$

and $f^2 - bc = \left[\frac{23}{2}\right]^2 - (-7)(-6) = \frac{529}{4} - 42 = \frac{529-168}{4} = \frac{361}{4} > 0 \Rightarrow f^2 > bc$

hence the given equation represents a pair of straight lines.

If θ is the acute angle between the lines $cos\theta = \frac{|a+b|}{\sqrt{(a-b)^2+4h^2}}$.

$\frac{|2-7|}{\sqrt{((2+7)^2+(-13)^2}} = \frac{5}{\sqrt{81+169}} = \frac{5}{5\sqrt{10}} = \frac{1}{\sqrt{10}} \Rightarrow \theta = cos^{-1}\left(\frac{1}{\sqrt{10}}\right)$

$\therefore$ Point of intersection = $\left[\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2}\right] = \left[\frac{19}{15}, \frac{7}{15}\right]$

9. Show that the given equation $2x^2 - xy - 6y^2 + 7y - 2 = 0$ represents a pair of straight lines.

Sol: Given pair of lines $2x^2 - xy - 6y^2 + 7y - 2 = 0$

Compare with a$x^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ then we get

$a = 2,\ b = -6,\ c = -2,\ \ h = \frac{1}{2},\ g = 0,\ f = \frac{7}{2}$

If $\Delta = 0$ then the above equation represents two pair of lines.

Now, $\Delta = abc + 2fgh - af^2 - bf^2 - ch^2$

$$= 2(-6)(-2) + 2\left[\frac{1}{2}\right](0)\left[\frac{1}{2}\right] - 2\left[\frac{7}{2}\right]^2 + 6[0]^2 + 2\left[\frac{1}{2}\right]^2 = 0$$

Also, $h^2 - ab = \left[\frac{1}{2}\right]^2 - 2(-6) = \frac{1}{4} + 12 = \frac{49}{4} > 0 \Rightarrow h^2 > ab$

and $g^2 - ac = [0]^3 - 2(-2) = 4 > 0 \Rightarrow g^2 > ac$

and $f^2 - bc = \left[\frac{7}{2}\right]^2 - (-6)(-2) = \frac{49}{4} - 12 = \frac{1}{4} > 0 \Rightarrow\ f^2 > bc$

Therefore, $h^2 > ab,\ g^2 > ac,\ f^2 > bc$

Hence the given equation presents a pair of straight lines

10. Show that the equation $3x^2 + 7xy + 2y^2 + 5x + 5y + 2 = 0$ represents a pair of straight lines. Find the point of intersection of the lines.

Sol: Given pair of lines $3x^2 + 7xy + 2y^2 + 5x + 5y + 2 = 0$

Compare with a$x^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ then we get

$a = 3,\ b = 2,\ c = 2,\ \ h = \frac{7}{2},\ g = \frac{5}{2},\ f = \frac{5}{2}$

If $\Delta = 0$ then the above equation represents two pair of lines.

Now, $\Delta = abc + 2fgh - af^2 - bf^2 - ch^2$

$$= 3(2)(2) + 2\left[\frac{5}{2}\right]\left[\frac{5}{2}\right]\left[\frac{7}{2}\right] - 3\left[\frac{5}{2}\right]^2 - 2\left[\frac{5}{2}\right]^2 - 2\left[\frac{7}{2}\right]^2 = 0$$

Also, $h^2 - ab = \left[\frac{7}{2}\right]^2 - 3(2) = \frac{49}{4} - 6 = \frac{49-24}{4} = \frac{25}{4} > 0 \Rightarrow h^2 > ab$

and $g^2 - ac = \left[\frac{5}{2}\right]^3 - 3(2) = \frac{25}{4} - 6 = \frac{25-24}{4} = \frac{1}{4} > 0 \Rightarrow g^2 > ac$

and $f^2 - bc = \left[\frac{5}{2}\right]^2 - (2)(2) = \frac{25}{4} - 4 = \frac{25-16}{4} = \frac{9}{4} > 0 \Rightarrow f^2 > bc$

Therefore, $h^2 > ab$, $g^2 > ac$, $f^2 > bc$

Hence the given equation presents a pair of straight lines

∴ Point of intersection = $\left[\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2}\right] = \left[\frac{-3}{5}, \frac{-1}{5}\right]$

MORE QUESTIONS FOR PRACTICE

11. Show that the equation $3x^2 - 8xy - 3y^2 = 0$, $3x^2 + 8xy - 3y^2 + 2x - 4y - 1 = 0$ forms a square.

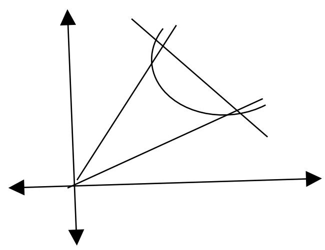
HOMOGENISATTION MODEL

12. Find the angle between the lines joining the origin to the points of intersection of the curve $x^2 + 2xy + y^2 + 2x + 2y - 5 = 0$ and the line $3x - y + 1 = 0$.

Sol: Given curve $x^2 + 2xy + y^2 + 2x + 2y - 5 = 0$ --------- (1) and the line $3x - y + 1 = 0$

$\Rightarrow y - 3x = 1$ -------- (2)

Homogenising equation (1) with (2), we obtain



$x^2 + 2xy + y^2 + 2x(1) + 2y(1) - 5(1)^2 = 0$

$\Rightarrow x^2 + 2xy + y^2 + 2x(y-3x) + 2y(y-3x) - 5(y-3x)^2 = 0$

$\Rightarrow x^2 + 2xy + y^2 + 2xy - 6x^2 + 2y^2 - 6xy - 5(y^2 - 6xy + 9x^2) = 0$

$\Rightarrow x^2 + 2xy + y^2 + 2xy - 6x^2 + 2y^2 - 6xy - 5y^2 + 30xy - 45x^2 = 0$

$\Rightarrow -50x^2 + 28xy - 2y^2 = 0 \Rightarrow 25x^2 - 14xy + y^2 = 0$

Compare with a$x^2 + 2hxy + by^2 = 0$ then we get $a = 25,\ 2h = -14,\ b = 1$

If θ is the acute angle between the pair of lines then

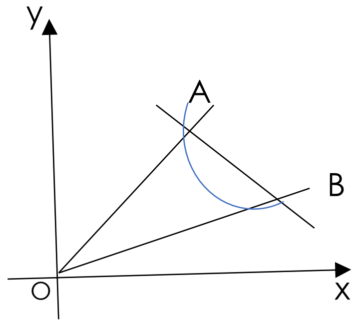
$cos\theta = \frac{|a+b|}{\sqrt{(a-b)^2+4h^2}}. = \frac{|25+1|}{\sqrt{(25-1)^2+(-14)^2}} = \frac{26}{\sqrt{576+196}} = \frac{26}{\sqrt{772}} = \frac{26}{2\sqrt{193}}.$

∴ $cos\theta = \frac{13}{\sqrt{193}} \Rightarrow \theta = cos^{-1}\left[\frac{13}{\sqrt{193}}\right]$

13. Show that the lines joining the origin to the points of intersection of the curve $x^2 - xy + y^2 + 3x + 3y - 2 = 0$ and line $x - y - \sqrt{2} = 0$ are mutually perpendicular.

Sol: Given curve $x^2 - xy + y^2 + 3x + 3y - 2 = 0$ ------- (1)

Given line $x - y - \sqrt{2} = 0 \Rightarrow x - y = \sqrt{2} \Rightarrow \frac{x-y}{\sqrt{2}} = 1$ ------ (2)



Homogenising equation (1) with (2), we obtain

$x^2 - xy + y^2 + 3x(1) + 3y(1) - 2(1)^2 = 0$

$\Rightarrow \frac{\sqrt{2}(x^2-xy+y^2)+3x(x-y)+3y(x-y)-\sqrt{2}(x-y)^2}{\sqrt{2}} = 0$

$\Rightarrow \frac{\sqrt{2}x^2-\sqrt{2}xy+\sqrt{2}y^2+3x^2-3xy-3y^2+3xy-\sqrt{2}\,x^2-\sqrt{2}y^2+2\sqrt{2}xy}{\sqrt{2}}$

$\Rightarrow 3x^2 + \sqrt{2}xy - 3y^2 = 0$

Comparing with a$x^2 + 2hxy + by^2 = 0$ then we get $a = 3,\ b = -3$

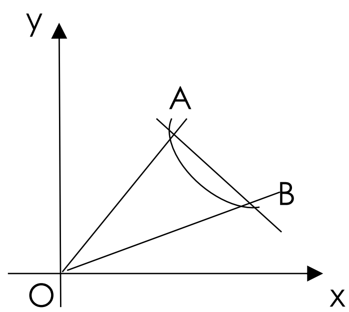
Clearly $a + b = 3 - 3 = 0$

∴ The lines $\overline{OA}$, $\overline{OB}$ are perpendicular.

14. Find the value of K, if the lines joining the origin to the points of intersection of the $2x^2 - 2xy + 3y^2 + 2x - y - 1 = 0$ and the line $x + 2y = K$are mutually perpendicular.

Sol: Given curve $2x^2 - 2xy + 3y^2 + 2x - y - 1 = 0$ -------- (1)

Given line $x + 2y = K \Rightarrow \frac{x+2y}{K} = 1$ ------ (2)



Homogenising equation (1) with (2), we obtain

$2x^2 - 2xy + 3y^2 + 2x(1) - y(1) - 1(1)^2 = 0$

$\Rightarrow 2x^2 - 2xy + 3y^2 + 2x\left(\frac{x+2y}{K}\right) - y\left(\frac{x+2y}{K}\right) - \left(\frac{x+2y}{K}\right)^2 = 0$

$\Rightarrow \frac{K^2(2x^2-2xy+3y^2)+2Kx(x+2y)-Ky(x+2y)-(x^2+4y^2+4xy)}{K^2} = 0$

$\Rightarrow \frac{2K^2x^2-2K^2xy+3K^2y^2+2Kx^2+4Kxy-Kxy-2Ky^2-x^2-4y^2-4xy}{K^2} = 0$

$\Rightarrow (2K^2 + 2K - 1)x^2 + (-2K^2 + 3K - 4)xy + (3K^2 - 2K - 4)y^2 = 0$

Compare with a$x^2 + 2hxy + by^2 = 0$ then we get

$a = 2K^2 + 2K - 1,\ b = 3K^2 - 2K - 4$

The lines are perpendicular So, $a + b = 0$

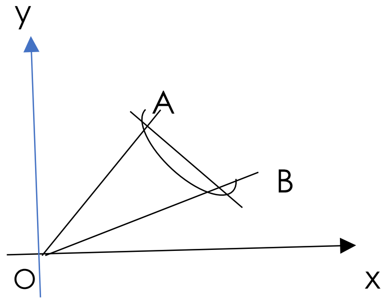
$\Rightarrow 2K^2 + 2K - 1 + 3K^2 - 2K - 4 = 0 \Rightarrow 5K^2 - 5 = 0$

$\Rightarrow 5K^2 = 5 \Rightarrow K^2 = 1 \Rightarrow K = \pm 1$

15. Show that the lines joining the origin to the points of intersection of the curve $7x^2 - 4xy + 8y^2 + 2x - 4y - 8 = 0$ with the line $3x - y = 2$ are mutually perpendicular.

Sol: Given curve $7x^2 - 4xy + 8y^2 + 2x - 4y - 8 = 0$ -------- (1)

Given line $3x - y = 2 \Rightarrow \frac{3x-y}{2} = 1$ ------ (2)



Homogenising equation (1) with (2), we obtain

$$7x^2 - 4xy + 8y^2 + 2x(1) - 4y(1) - 8(1)^2 = 0$$

$\Rightarrow 7x^2 - 4xy + 8y^2 + 2x\left(\frac{3x-y}{2}\right) - 4y\left(\frac{3x-y}{2}\right) - 8\left(\frac{3x-y}{2}\right)^2 = 0$

$\Rightarrow\ 7x^2 - 4xy + 8y^2 + 2x(3x - y) - 2y(3x - y) - 2(9x^2 - 6xy + y^2)$

$\Rightarrow 7x^2 - 4xy + 8y^2 + 3x^2 - xy - 6xy + 2y^2 - 18x^2 - 2y^2 + 12xy = 0$

$\Rightarrow -8x^2 + xy + 8y^2 = 0 \Rightarrow 8x^2 - xy - 8y^2 = 0$

Compare with $ax^2 + 2hxy + by^2 = 0$ then we get $a = 8,\ b = -8$

Clearly $a + b = 8 - 8 = 0$

$\therefore$ The lines $\overline{OA}, \overline{OB}$ are mutually perpendicular.

16. Find the angle between the pair of lines represented by the equation $x^2 - 7xy + 12y^2 = 0$

Sol: Given pair of lines $x^2 - 7xy + 12y^2 = 0$

Compare with $ax^2 - 2hxy + by^2 = 0$ we get $a = 1,\ 2h = -7, b = 12$

Let θ be the angle between the pair of lines

$cos\theta = \frac{|a+b|}{\sqrt{(a-b)^2+(2h)^2}} . = \frac{|1+12|}{\sqrt{(1-12)^2+(-7)^2}} = \frac{13}{\sqrt{121+49}} = \frac{13}{\sqrt{170}}.$

$$\therefore cos\theta = \frac{13}{\sqrt{170}} \Rightarrow \theta = cos^{-1}\left[\frac{13}{\sqrt{170}}\right]$$

17. Write drawn the equations of the pair of straight lines joining the origin to the points of intersection of the line $6x - y + 8 = 0$ with the pair of straight lines $3x^2 + 4xy - 4y^2 - 11x + 2y + 6 = 0$. Show that the lines so obtained make equal angles with the coordinate axes.

Sol: Given curve $3x^2 + 4xy - 4y^2 - 11x + 2y + 6 = 0$ -------- (1)

Given line $6x - y + 8 = 0 \Rightarrow y - 6x = 8 \Rightarrow \frac{y-6x}{8} = 1$ ------ (2)

Homogenising equation (1) with (2), we obtain

$$3x^2 + 4xy - 4y^2 - 11x(1) - 2y(1) + 6(1)^2 = 0$$

$\Rightarrow 3x^2 + 4xy - 4y^2 - 11x\left(\frac{y-6x}{8}\right) - 2y\left(\frac{y-6x}{8}\right) + 6\left(\frac{y-6x}{8}\right)^2 = 0$

$\Rightarrow 32(3x^2 + 4xy - 4y^2) + 4(2y - 11x)(y - 6x) + 3(y - 6x)^2 = 0$

$\Rightarrow 96x^2 + 128xy - 128y^2 + 8y^2 - 92xy + 264x^2 + 3y^2 - 36xy + 108x^2 = 0$

$\Rightarrow 468x^2 - 117y^2 = 0 \Rightarrow 4x^2 - y^2 = 0$

Equation of the pair of angle bisector of $4x^2 - y^2 = 0$ is

$0(x^2 - y^2) - (4 + 1)xy = 0 \Rightarrow =xy = 0$

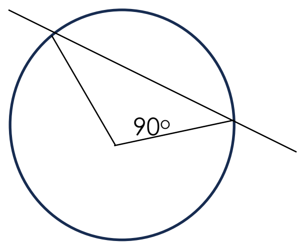
∴ The lines represented by $4x^2 - y^2 = 0$ are equally inclined the coordinate axes.

18. Find the condition for the chord $lx + my = 1$ of the circle $x^2 + y^2 = a^2$ (whose centre is the origin) to subtend a right angle at the origin.

Sol: Given circle $x^2 + y^2 = a^2$ ------ (1)

Given line $lx + my = 1$ -------- (2)

Homogenising equation (1) with (2), we obtain



$x^2 + y^2 = a^2(1)^2 \Rightarrow x^2 + y^2 = a^2(lx + my)^2$

$\Rightarrow\ x^2 + y^2 = a^2(l^2m^2 + m^2y^2 + 2lxmy)$

$\Rightarrow a^2l^2x^2 + a^2m^2y^2 + 2a^2lxmy - x^2 - y^2 = 0$

$\Rightarrow (a^2l^2 - 1)x^2 + 2a^2lmxy + (a^2m^2 - 1)y^2 = 0$

Compare with $Ax^2 + 2Hxy + By^2 = 0$

Then we get $A = a^2l^2 - 1,\ H = a^2lm, B = a^2m^2 - 1$

Since the lines are right angle at the origin, we have A + B = 0

$\Rightarrow a^2l^2 - 1 + a^2m^2 - 1 = 0$

$\Rightarrow a^2l^2 + a^2m^2 - 2 = 0\ \Rightarrow a^2(l^2 + m^2) = 2$

19. Find the condition for the lines joining the origin to the points of intersection of the circle $x^2 + y^2 = a^2$ and the line $lx + my = 1$to coincide.

Sol: Given circle $x^2 + y^2 = a^2$ ------ (1)

Given line $lx + my = 1$ -------- (2)

Homogenising equation (1) with (2), we obtain

$x^2 + y^2 = a^2(1)^2 \Rightarrow x^2 + y^2 = a^2(lx + my)^2$

$\Rightarrow\ x^2 + y^2 = a^2(l^2m^2 + m^2y^2 + 2lxmy)$

$\Rightarrow a^2l^2x^2 + a^2m^2y^2 + 2a^2lxmy - x^2 - y^2 = 0$

$\Rightarrow (a^2l^2 - 1)x^2 + 2a^2lmxy + (a^2m^2 - 1)y^2 = 0$

Compare with $Ax^2 + 2Hxy + By^2 = 0$

Then we get $A = a^2l^2 - 1,\ H = a^2lm, B = a^2m^2 - 1$

Since the lines are right angle at the origin, we have A + B = 0

$\Rightarrow a^2l^2 - 1 + a^2m^2 - 1 = 0$

$\Rightarrow a^2l^2 + a^2m^2 - 2 = 0\ \Rightarrow a^2(l^2 + m^2) = 2$

LEVEL – 2 (7 Marks)

1. Show that the product of the perpendicular distance from the origin to the pair of straight lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is $\frac{|c|}{\sqrt{(a-b)^2+4h^2}}$.

Sol: Let $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents the pair of lines

$l_1x + m_1y + n_1 = 0$ -------- (1)

$l_2x + m_2y + n_2 = 0$ ---------- (2)

$\therefore l_1l_2 = a,\ l_1m_2 + l_2m_1 = 2h,\ m_1m_2 = b,\ n_1n_2 = c$

The length of perpendicular from (0, 0) to the line (1)

$$= \frac{|l_1(0)+m_1(0)+n_1|}{\sqrt{l_1^2+m_1^2}} = \frac{|n_1|}{\sqrt{l_1^2+m_1^2}}$$

The length of perpendicular from (0, 0) to the line (2)

$$= = \frac{|l_2(0)+m_2(0)+n_2|}{\sqrt{l_2^2+m_2^2}} = \frac{|n_2|}{\sqrt{l_2^2+m_2^2}}$$

$\therefore$ Product of perpendicular = $\frac{|n_1|}{\sqrt{l_1^2+m_1^2}} \times \frac{|n_2|}{\sqrt{l_2^2+m_2^2}}$

$$= \frac{|n_1n_2|}{\sqrt{l_1^2l_2^2+l_1^2m_2^2+m_1^2l_2^2+m_1^2m_2^2}} = \frac{|n_1n_2|}{\sqrt{(l_1l_2)^2+(l_1m_2+l_2m_1)^2-2l_1l_2m_1m_2+(m_1+m_2)^2}}$$

$$= \frac{|c|}{\sqrt{a^2+(2h)^2-2ab+b^2}} = \frac{|c|}{\sqrt{(a-b)^2+4h^2}}$$

2. If the point of intersection of the pair of lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and hence deduce that $abc + 2fgh - af^2 - bf^2 - ch^2$ from above.

Sol: Given pair of lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Let $P(x_1, y_1)$ be the point of intersection of the pair of lines shifting the origin to $P(x_1, y_1)$ by translation of axes, the equation changes to.

$a(x + x_1)^2 + 2h(x + x_1)(y + y_1) + b(y + y_1)^2 + 2g(x + x_1) + 2f(y + y_1) + c = 0$

$\Rightarrow a(x^2 + x_1^2 + 2xx_1) + 2h(xy + xy_1 + x_1y + x_1y_1) + b(y^2 + y_1^2 + 2yy_1) + 2gx + 2gx_1 + 2fy + 2fy_1 + c = 0$

$\Rightarrow (ax^2 + 2hxy + by^2) + 2(ax_1 + hy_1 + g)x + 2(hx_1 + by_1 + f)y +$

$(ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c) = 0$

This equation represents pair of lines passing through origin when

$ax_1 + hy_1 + g = 0$ ------ (1) and $hx_1 + by_1 + f = 0$ ------- (2)

Solving (1) & (2)

$x \quad y \quad 1 \qquad \Rightarrow \frac{x_1}{hf-bg} = \frac{y_1}{hf-bg} = \frac{1}{ab-h^2}$

h g a h

b f h b

$\Rightarrow x_1 = \frac{hf-bg}{ab-h^2}, \quad y_1 = \frac{gh-af}{ab-h^2}$

∴ Point of intersection (x_1, y_1) = $\left[\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2}\right]$

We have, $ax_1^2 + 2hx_1y_1 + 2gx_1 + 2fy_1 + c = 0$

$\Rightarrow (ax_1 + by_1 + g)x_1 + (nx_1 + by_1 + f)y_1 + (gx_1 + fy_1 + c) = 0$

$\Rightarrow (0)x + (0)y + (gx_1 + fy_1 + c) = 0$

But, point of intersection $\left[\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2}\right]$ satisfies $(gx_1 + fy_1 + c) = 0$

$\Rightarrow g\left[\frac{hf-bg}{ab-h^2}\right] + f\left[\frac{gh-af}{ab-h^2}\right] + c = 0$

$\Rightarrow g(hf - bg) + f(gh - af) + c(ab - h^2) = 0$

$\Rightarrow ghf - bg^2 + ghf - af^2 + abc - ch^2 = 0$

$\Rightarrow abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

3. Find the centroid and area of the triangle formed by the lines $3x^2 - 4xy + y^2 = 0$ and $2x - y = 6$.

Sol: Given lines are $3x^2 - 4xy + y^2 = 0$

$\Rightarrow 3x^2 - 4xy + y^2 = 0 \Rightarrow 3x^2 - 3xy - xy + y^2 = 0$

$\Rightarrow 3x(x - y) - y(x - y) = 0 \Rightarrow (x - y)(3x - y) = 0$

$\Rightarrow x - y = 0$ ---- (1) $3x - y = 0$ -------(2)

Solving (1) & (2), we get O(0, 0)

Solving (2) & (3) we get A

	x	y	1	$\Rightarrow \frac{x}{6+0} = \frac{y}{0+18} = \frac{1}{-3+2}$
-1	0	3	-1	$\Rightarrow \frac{x}{6} = \frac{y}{18} = \frac{1}{-1}$
-1	-6	2	-1	$\Rightarrow x = -6,\ y = -18$

∴ Point of intersection A(-6, -18)

Solving (3) & (1) we get B

	x	y	1	$\Rightarrow \frac{x}{0-6} = \frac{y}{-6-0} = \frac{1}{-2+1}$
-1	-6	2	-1	$\Rightarrow \frac{x}{-6} = \frac{y}{-6} = \frac{1}{-1}$
-1	0	1	-1	$\Rightarrow x = 6,\ y = 6$

∴ Point of intersection B(6, 6)

∴ Centroid of ΔOAB = $\left[\frac{0-6+6}{3}, \frac{0-18+6}{3}\right] = \left[\frac{0}{3}, \frac{-12}{3}\right] = (0, 4)$

Area of $\Delta OAB = \frac{1}{2}|x_1y_2 - x_2y_1| = \frac{1}{2}|(-6)(6) - 6(-18)|$

$= \frac{1}{2}|-36 + 108| = \frac{1}{2}|72|$ = 36 sq. units

4. Find the centroid and area of the triangle formed by the lines $2y^2 - xy - 6x^2 = 0$ and $x + y + 4 = 0$.

Sol: Given lines are $2y^2 - xy - 6x^2 = 0$ ------ (1) and $x + y + 4 = 0$ ------ (2)

(2) $\Rightarrow y = -(x + y)$

(1) $\Rightarrow 2(x + y)^2 + x(x + 4) - 6x^2 = 0 \Rightarrow 2(x^2 + 16 + 8x) + x^2 + 4x - 6x^2 = 0$

$\Rightarrow 2x^2 + 32 + 16x + x^2 + 4x - 6x^2 = 0 \Rightarrow -3x^2 + 20x + 32 = 0$

$\Rightarrow 3x^2 - 20x - 32 = 0 \Rightarrow 3x^2 - 24x + 4x - 32 = 0$

$\Rightarrow 3x(x - 8) + 4(x - 8) = 0 \Rightarrow (x - 8)(3x + 4) = 0 \Rightarrow x = 8,\ \frac{-4}{3}$

$x = 8 \Rightarrow y = -(8 + 4) \Rightarrow y = -12$

∴ Point of intersection A(8, -12)

$x = -\frac{4}{3} \Rightarrow y = -\left[-\frac{4}{3} + 4\right] = \frac{-8}{3}$

∴ Another point of intersection B$\left[-\frac{4}{3}, \frac{-8}{3}\right]$

Also the pair of line $2y^2 - xy - 6x^2 = 0$ intersect at O(0, 0)

$\therefore$ Centroid of ΔOAB= $\left[\frac{0+8-\frac{4}{3}}{3}, \frac{0-12-\frac{8}{3}}{3}\right] = \left[\frac{24-4}{9}, \frac{-36-8}{9}\right] = \left[\frac{20}{9}, \frac{-44}{9}\right]$

Area of $\Delta OAB = \frac{1}{2}|x_1y_2 - x_2y_1| = \frac{1}{2}\left|8\left(-\frac{8}{3}\right) - \left(-\frac{4}{3}\right)(-12)\right|$

$= \frac{1}{2}\left|-\frac{64}{3} - 16\right| = \frac{1}{2}\left|\frac{-64-48}{3}\right| = \frac{112}{6} = \frac{56}{3}$ sq. units

5. Find the centroid of the triangle formed by $12x^2 - 20xy + 7y^2 = 0$ and $2x - 3y + 4 = 0$.

Sol: Given pair of lines is $12x^2 - 20xy + 7y^2 = 0$ -------- (1)

and $2x - 3y + 4 = 0$ -------- (2)

(2) $\Rightarrow 2x = 3y - 4$

(1) $\Rightarrow 12x^2 - 20xy + 7y^2 = 0 \Rightarrow 3(2x)^2 - 10y(2x) + 7y^2 = 0$

$\Rightarrow 3(3y-4)^2 - 10y(3y-4) + 7y^2 = 0 \Rightarrow 3(9y^2 + 16 - 24y) - 30y^2 + 40y + 7y^2 = 0$

$\Rightarrow 27y^2 - 48 - 72y - 30y^2 + 40y + 7y^2 = 0$

$\Rightarrow 4y^2 - 32y + 48 = 0 \Rightarrow y^2 - 8y + 12 = 0 \Rightarrow (y-2)(y=6) = 0 \Rightarrow y = 2, 6$

$\therefore (2) \Rightarrow 2x = 3(2) - 4 \Rightarrow x = 1 \text{ and } 2x = 3(6) - 4 \Rightarrow x = 7$

$\therefore$ The point of intersections are A(1, 2) and B(7, 6) and O(0, 0)

$\therefore$ Centroid of ΔOAB is $\left[\frac{0+1+7}{3}, \frac{0+2+6}{3}\right] = \left[\frac{8}{3}, \frac{8}{3}\right]$

6. If the equation $mx^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$ represents a pair of straight lines, find 'm' and also find the angle and point of intersection for this value of 'm'.

Sol: Given pair of lines $mx^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$

Compare with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ then we get

$a = m,\ b = 12,\ c = -3,\ h = -5,\ g = \frac{5}{2}, f = -8$

If $\Delta = 0$ then the above equation represents two pair of lines

Now, $\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$\Rightarrow m(12)(-3) + 2(-8)\left(\frac{5}{2}\right)(-5) - m(-8)^2 - 12\left(\frac{5}{2}\right)^2 + 3(-5)^2 = 0$

$\Rightarrow -36m + 200 - 64m - 75 + 75 = 0$

$\Rightarrow 100m = 200 \Rightarrow m = 2$ hence $a = 2$

If θ is the acute angle between the lines then

$cos\theta = \frac{|a+b|}{\sqrt{(a-b)^2+4h^2}}. = \frac{14}{\sqrt{100+100}} = \frac{14}{\sqrt{200}} = \frac{7}{\sqrt{50}}.$

$cos\theta = \frac{7}{\sqrt{50}} \Rightarrow \theta = cos^{-1}\left[\frac{7}{\sqrt{50}}\right]$

∴ Point of intersection = $\left[\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2}\right]$ = $\left[-10, -\frac{7}{2}\right]$

7. Theorem 7 : If (α, β) is the centroid of the triangle formed $ax^2 + 2hxy + by^2 = 0$ and the line $lx + my = 1$, then prove that $\frac{\alpha}{bl-hm} = \frac{\beta}{am-hl} = \frac{2}{3(am^2-2hlm+bl^2)}$.

Sol: Let the equation $ax^2 + 2hxy + by^2 = 0$ represents the lines

$l_1x + m_1y = 0$ ------------ (1)

$l_2x + m_2y = 0$ ----------- (2)

$x^2 + 2hxy + by^2 = (l_1x + m_1y)(l_2x + m_2y)$

$= l_1l_2x^2 + (l_1m_2 + l_2m_1)xy + m_1m_2y^2$

Comparing on both sides we get

$a = l_1l_2,\ 2h = l_1m_2 + l_2m_1,\ b = m_1m_2$

Let the given line $lx + my - 1 = 0$ --------- (3)

Solving eq (1) & (2) we get O(0, 0) = (x_1, y_1)

Solving eq (1) & (3) we get A:

x	y	1	
m_1	0	l_1	m_1
m	-1	l	m

$\frac{x}{-m_1-0} = \frac{y}{0 \mp l_1} = \frac{1}{l_1m - lm_1}$

$\Rightarrow x = \frac{-m_1}{l_1m-lm_1}, y = \frac{l_1}{l_1m-lm_1}$

$\therefore A = \left(\frac{-m_1}{l_1m-lm_1}, \frac{l_1}{l_1m-lm_1}\right) = (x_2, y_2)$

Similarly solving (2) & (3) we get (B) $\therefore B = \left(\frac{-m_2}{l_2m-lm_2}, \frac{l_2}{l_2m-lm_2}\right) = (x_3, y_3)$

Centroid of ΔOAB : $(\alpha, \beta) = \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

$$= \left(\frac{0+\frac{-m_1}{l_1m-lm_1}+\frac{l_1}{l_1m-lm_1}}{3}, \frac{0+\frac{-m_2}{l_2m-lm_2}+\frac{l_2}{l_2m-lm_2}}{3}\right)$$

$\therefore \alpha = \frac{1}{3}\left[\frac{-m_1}{l_1m-lm_1} + \frac{-m_2}{l_2m-lm_2}\right]$ and $\beta = \frac{1}{3}\left[\frac{l_1}{l_1m-lm_1} + \frac{l_2}{l_2m-lm_2}\right]$

$\Rightarrow \alpha = \frac{1}{3}\left[\frac{-m_1(l_2m-lm_2)-m_2(l_1m-lm_1)}{(l_1m-lm_1)(l_2m-lm_2)}\right]$ and $\beta = \frac{1}{3}\left[\frac{l_1(l_2m-lm_2)\mp(l_1m-lm_1)}{(l_1m-lm_1)(l_2m-lm_2)}\right]$

$\Rightarrow \alpha = \frac{1}{3}\left[\frac{-l_2m_1m+lm_1m_2-l_1m_2m+lm_1m_2}{l_1l_2m^2-ll_1mm_2-ll_2mm_1+m_1m_2l^2}\right]$ and $\beta = \frac{1}{3}\left[\frac{l_1l_2m-ll_1m_2+l_1l_2m-ll_2m_1}{l_1l_2m^2-ll_1mm_2-ll_2mm_1+m_1m_2l^2}\right]$

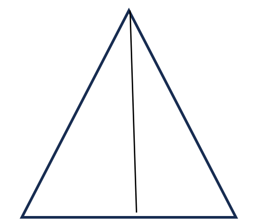
$\Rightarrow \alpha = \frac{1}{3}\left[\frac{2m_1m_2l-(l_1m_2+l_2m_1)m}{l_1l_2m^2-(l_1m_2+l_2m_1)lm+m_1m_2l^2}\right]$ and $\beta = \frac{1}{3}\left[\frac{2l_1l_2m-(l_1m_2+l_2m_1)l}{l_1l_2m^2-(l_1m_2+l_2m_1)lm+m_1m_2l^2}\right]$

$\Rightarrow \alpha = \frac{1}{3}\left[\frac{2bl-2hm}{am^2-2hlm+bl^2}\right], \ \beta = \frac{1}{3}\left[\frac{2am-2hl}{am^2-2hlm+bl^2}\right]$

$= \frac{\alpha}{bl-hm} = \frac{2}{3(am^2-2hlm+bl^2)}, \ \frac{\beta}{am-hl} = \frac{2}{3(am^2-2hlm+bl^2)}$

$\therefore \frac{\alpha}{bl-hm} = \frac{\beta}{am-hl} = \frac{2}{3(am^2-2hlm+bl^2)}$

8. Show that lines represented by $(lx+my)^2 - 3(mx-ly)^2 = 0$ and $lx+my+n=0$ form an equilateral triangle with area $\frac{n^2}{\sqrt{3}(l^2+m^2)}$.



Sol: Given pair of lines are $(lx+my)^2 - 3(mx-ly)^2 = 0$

$\Rightarrow (lx+my)^2 - \left[\sqrt{3}(mx-ly)\right]^2 = 0$

$\Rightarrow \left[lx+my+\sqrt{3}(mx-ly)\right]\left[lx+my-\sqrt{3}(mx-ly)\right]$

$\Rightarrow (l+\sqrt{3}m)x + (m-\sqrt{3}l)y = 0$ ------- (1) and $(l-\sqrt{3}m)x + (m+\sqrt{3}l)y = 0$ ------- (2)

Also given another line $lx+my+n=0$ -------- (3)

If A is the angle between (1) & (3) then

$\text{Cos}\,A = \frac{(l+\sqrt{3}m)l+(m-\sqrt{3}l)m}{\sqrt{(+\sqrt{3}m)^2+(m-\sqrt{3}l)^2}\sqrt{l^2+m^2}} = \frac{l^2+\sqrt{3}lm+m^2-\sqrt{3}lm}{\sqrt{l^2+3m^2+2\sqrt{3}lm+m^2+3l^2-2\sqrt{3}lm\sqrt{l^2+m^2}}}$

$= \frac{l^2+m^2}{\sqrt{4l^2+4m^2\sqrt{l^2+m^2}}} = \frac{l^2+m^2}{2(l^2+m^2)} = \frac{1}{2} \qquad \therefore cosB = \frac{1}{2} \Rightarrow B = 60^o$

Also the lines (1), (2), (3) intersects pair wise so, angle between (1) & (2) is

180° – (60° + 60°) = 60°

$\therefore$ ΔOAB is equilateral

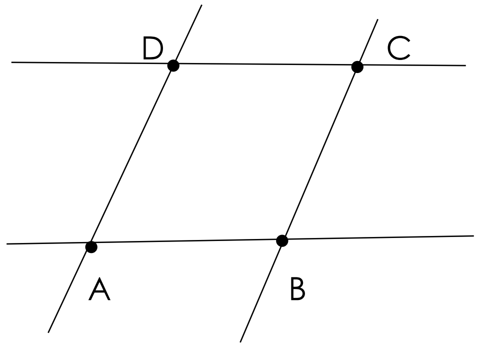
Let 'A' be the altitude through O(0, 0) to the line $lx+my+n=0$ then

$h = \frac{|l(0)+m(0)+n|}{\sqrt{l^2+m^2}} = \frac{|n|}{\sqrt{l^2+m^2}}$

∴ Area of ΔOAB = $\frac{h^2}{\sqrt{3}} = \frac{h^2}{\sqrt{3}(l^2+m^2)}$

9. Show that the pairs of lines $y^2 - 4y + 3 = 0$, $x^2 + 4xy + 4y^2 + 5x + 10y + 4 = 0$ form a parallelogram. Also find the length of its sides.

Sol: Given equation is $x^2 + 4xy + 4y^2 + 5x + 10y + 4 = 0$



$\Rightarrow (x + 2y)^2 + 5x + 10y + 4 = 0$

Take $x^2 + 4xy + 4y^2 + 5x + 10y + 4 = 0$

= $(x + 2y + l)(x + 2y + m)$

= $x^2 + 4xy + 4y^2 + (l + m)x + (2l + 2m)y + lm$

Comparing on the sides we get

$l + m = 5,\ lm = 4 \ \Rightarrow m = \frac{4}{l}$

$\Rightarrow l + \frac{4}{l} = 5 \Rightarrow \frac{l^2+4}{l} = 5$

$\Rightarrow l^2 + 4 = 5l \Rightarrow l^2 - 5l + 4 = 0 \Rightarrow (l - 1)(l - 4) = 0$

Take $l = 1 \Rightarrow m = 4$

∴ The lines are $x + 2y + 1 = 0$ ------ (1) and $x + 2y + 4 = 0$ -------- (2)

Also, $y^2 - 4y + 3 = 0 \Rightarrow (y - 3)(y - 1) = 0$

$\Rightarrow y = 3$ ----- (3) and $y = 1$ ------- (4)

The lines (1) and (2) are parallel, and (3) & (4) are parallel

∴ The given lines form a parallelogram

Solving (1) & (3) we get (1) $\Rightarrow x + 2(3) + 1 = 0 \Rightarrow x = -7$

∴ A = (-7, 3)

Solving (1) & (4) we get (2) $\Rightarrow x + 2(1) + 1 = 0 \ \Rightarrow x = -3$

∴ B = (-3, 1)

Solving (2) & (4) we get (1) ⇒ $x + 2(1) + 4 = 0 \Rightarrow x = -6$

∴ AC = (-7, 3)

Solving (2) & (3) we get (2) ⇒ $x + 2(3) + 4 = 0 \Rightarrow x = -10$

∴ D = (-10, 3)

Length of the sides

BC = $\sqrt{(-6+3)^2 + (1-1)^2} = 3$

AB = $\sqrt{(-7+3)^2 + (3-1)^2} = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$

CHAPTER 5 : 3D COORDINATES

KEY POINTS

- Distance between O(0, 0, 0), P(x, y, z) is OP $=\sqrt{x^2+y^2+z^2}$
- Distance between $A(x_1, y_1, z_1)$, $B(x_1, y_2, z_2)$ is AB = $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$
- If a point P divides the line segment joining $A(x_1, y_1, z_1)$, $B(x_1, y_2, z_2)$ in the ratio m : n internally then

  $\left[\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}, \frac{mz_2+nz_1}{m+n}\right]$
- The ratio in which the point $P(x,, y, z)$ divides the line segment joining $A(x_1, y_1, z_1)$, $B(x_1, y_2, z_2)$ is

  $x_1 - x : x - x_2$ (or) $y_1 - y : y - y_2$ (or) $z_1 - z : z - z_2$
- The line segment joining (x_1, y_1, z_1) and (x_1, y_2, z_2) is divided by

  i) yz-plane in the ratio $-x_1 : x_2$

  ii) zx – plane in the ratio $-y_1 : y_2$

  iii) xy – plane in the ratio $-z_1 : z_2$
- Three points $A(x_1, y_1, z_1)$, $B(x_1, y_2, z_2)$, $C(x_3, y_3, z_3)$ are collinear then

  (i) $AB + BC = AC$ (OR) ii) $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$
- Centroid of ΔABC with vertices $A(x_1, y_1, z_1)$, $B(x_1, y_2, z_2)$, $C(x_3, y_3, z_3)$ is

  $G = \left[\frac{x_1+x_2+x_3}{3} + \frac{y_1+y_2+y_3}{3} + \frac{z_1+z_2+z_3}{3}\right],$
- Centroid of the tetrahedron with vertices $A(x_1, y_1, z_1)$, $B(x_1, y_2, z_2)$, $C(x_3, y_3, z_3)$, $D(x_4, y_4, x_4)$ is

  $G = \left[\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4}\right]$

LEVEL 1 (2Marks)

1. Find x, if the distance between (5, -1, 7) and (x, 5, 1) is 9 units.

Sol: Let A (5, -1, 7) and B(x, 5, 1)

Now AB = $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

= $\sqrt{(x-5)^2+(5+1)^2+(1-7)^2}$

= $\sqrt{(x-5)^2+36+36}$ = $\sqrt{(x-5)^2+72}$

But AB = 9 ⇒ AB^2 = 81

⇒ $(x-5)^2+72$ = 81

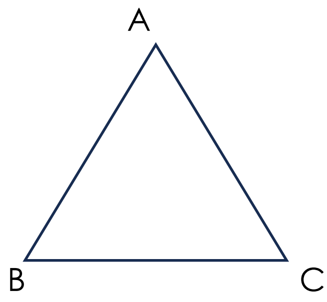
⇒ $(x-5)^2$ = 9 ⇒ $x-5=\pm 3$

⇒ $x=3+5$ (or) $x=-3+5$

∴ $x=8,2$

2. Show that the points (1, 2, 3), (2, 3, 1), (3, 1, 2) form an equilateral triangle.

Sol: Let the given points are A (1, 2, 3), B (2, 3, 1), C(3, 1, 2)



AB = $\sqrt{(2-1)^2+(3-2)^2+(1-3)^2}$ = $\sqrt{1+1+4}$ = $\sqrt{6}$

BC = $\sqrt{(3-2)^2+(1-3)^2+(2-1)^2}$ = $\sqrt{1+4+1}$ = $\sqrt{6}$

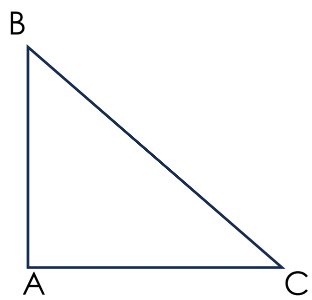
CA = $\sqrt{(3-1)^2+(1-2)^2+(2-3)^2}$ = $\sqrt{4+1+1}$ = $\sqrt{6}$

Clearly AB = BC = AC

Hence, given points form an equilateral triangle.

3. Show that the points (2, 3, 5), (-1, 5, -1) and (4, -3, 2) form a right angles isosceles triangle.

Sol: Let A (2, 3, 5), B (-1, 5, -1), C (4, -3, 2)



AB = $\sqrt{(2+1)^2+(3-5)^2+(5+1)^2}$ = $\sqrt{9+4+36}$ = $\sqrt{7}$

BC = $\sqrt{(-1-4)^2+(5+3)^2+(-1-2)^2}$ = $\sqrt{25+64+9}$ = $7\sqrt{2}$

CA = $\sqrt{(4-2)^2+(-3-3)^2+(2-5)^2}$ = $\sqrt{4+36+9}$ = $\sqrt{49}$ = 7

AB = CA and $AB^2 + AC^2 = 49 + 49 = 98 = BC^2$

∴ ΔABC is a right angled isosceles triangle.

4. Show that ABCD is a square where A, B, C, D are the points (0, 4, 1), (2, 3, -1), (4, 5, 0) and (2, 6, 2) respectively.

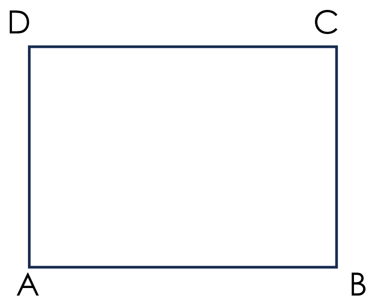
Sol: Let

AB $= \sqrt{(0-2)^2+(4-3)^2+(1+1)^2} = \sqrt{4+1+4} = 3$

BC $= \sqrt{(2-4)^2+(3-5)^2+(-1-0)^2} = \sqrt{4+4+1} = 3$

CD $= \sqrt{(3-1)^2+(1-2)^2+(2-3)^2} = \sqrt{4+1+1} = \sqrt{6}$

DA $= \sqrt{(2-0)^2+(6-4)^2+(2-1)^2} = \sqrt{4+4+1} = \sqrt{6}$



AC $= \sqrt{(4-0)^2+(5-4)^2+(0-1)^2} = \sqrt{16+1+1} = \sqrt{18}$

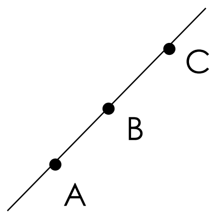
BD $= \sqrt{(2-2)^2+(3-6)^2+(-1-2)^2} = \sqrt{0+9+9} = \sqrt{18}$

Clearly AB = BC = CD = DA and AC = BD

∴ A, B, C, D forms a square.

5. Show that the points (1, 2, 3), 7, 0, 1), (-2, 3, 4) are collinear.

Sol: Let A (1, 2, 3), B(7, 0, 1), C(-2, 3, 4)



D. r's of $\overline{AB}$ are

(7 – 1, 0 – 2, 1- 3) = (6, -2, -2) = -2(-3, 1, 1)

D. r's are $\overline{AC}$ are

(-2-1, 3 – 2, 4 – 3) = (-3, 1, 1)

Clearly the d.r's of $\overline{AB}$ and $\overline{AC}$ are proportional.

So, given points are collinear.

6. Find the ratio in which yz – plane divides the lines joining A(2, 4, 5) and B(3, 5, -4). Also fin the point of intersection.

Sol: The ratio in which yz-plane divides the line joining A(x_1, y_1, z_1), $B(x_1, y_2, z_2)$ is

$-x_1 : x_2$ = -2 : 3

Point of intersection: Let

A(2, 4, 5) = A(x_1, y_1, z_1), B(3, 5, -4) = $B(x_1, y_2, z_2)$

The point which divides $\overline{AB}$ in the ratio -2 : 3 is $\left[\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}, \frac{mz_2+nz_1}{m+n}\right]$

= $\left[\frac{(-2)(3)+3(2)}{-2+3}, \frac{(-2)(5)+3(4)}{-2+3}, \frac{(-2)(-4)+3(5)}{-2+3}\right]$ = $\left[\frac{-6+6}{1}, \frac{-10+12}{1}, \frac{8+15}{1}\right]$ = [0, 2, 23]

7. Find the ratio in which xz-plane divides the line joining A(-2, 3, 4) and B(1, 2, 3).

Sol: Given that A(-2, 3, 4) and B(1, 2, 3)

The ratio in which xz-plane divides the line joining

A(x_1, y_1, z_1), $B(x_1, y_2, z_2)$ is $-y_1 : y_2$ = -3 : 2

8. Find the centroid of the triangle with vertices A(5, 4, 6), B(1, -1, 3), C(4, 3, 2).

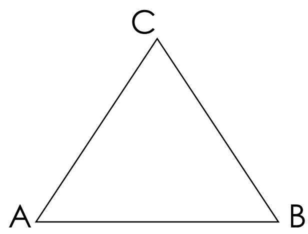
Sol: Centroid of ΔABC $\left[\frac{x_1+x_2+x_3}{3} + \frac{y_1+y_2+y_3}{3} + \frac{z_1+z_2+z_3}{3}\right]$,

= $\left[\frac{5+1+4}{3}, \frac{4-1+3}{3}, \frac{6+3+2}{3}\right]$ = $\left[\frac{10}{3}, 2, \frac{11}{3}\right]$

9. In ΔABC, the centroid is the origin and the vertices of A, B are (1, 1, 1) and (-2, 4, 1) respectively, then find C.

Sol: Given that A(1, 1, 1), B(-2, 4, 1) and take C(x, y, z)

Centroid of ΔABC = $\left[\frac{1-2+x}{3}, \frac{1+4+y}{3}, \frac{1+1+z}{3}\right]$



Given that $\left[\frac{x-1}{3}, \frac{y+5}{3}, \frac{z+2}{3}\right] = (0, 0, 0)$

$\Rightarrow \frac{x-1}{3} = 0 \Rightarrow x - 1 = 0 \Rightarrow x = 1$

$\Rightarrow \frac{y+5}{3} = 0 \Rightarrow y + 5 = 0 \Rightarrow y = -5,$

$\Rightarrow \frac{z+2}{3} = 0 \Rightarrow z + 2 = 0 \Rightarrow z = -2$

∴ third vertex C(x, y, z) = (1, -5, -2)

10. Find the centroid of tetrahedron vertices are (2, 3, -4), (-3, 3, -2), (-1, 4, 2), (3, 5, 1).

Sol: Let A (2, 3, -4), B (-3, 3, -2), C (-1, 4, 2), D (3, 5, 1)

Centroid of the tetrahedron ABCD is

= $\left[\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4}\right]$

$= \left[\frac{2-3-1+3}{4}, \frac{3+3+4+5}{4}, \frac{-4\pm2+2+1}{4}\right]$

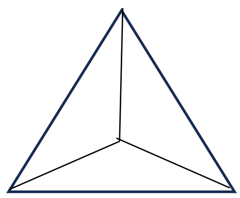
$= \left[\frac{1}{4}, \frac{15}{4}, \frac{-3}{4}\right]$

11. If (3, 2, -1), (4, 1, 1) and (6, 2, 5) are three vertices and (4, 2, 2) is the centroid of a tetrahedron. Find the fourth vertex.

Sol: Given A(3, 2, -1), B(4, 1, 1,), C(6, 2, 5)

Let D(x, y, z)

Centroid of the tetrahedron ABCD is



$= \left[\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4}\right]$

$= \left[\frac{3+4+6+x}{4}, \frac{2+1+2+y}{4}, \frac{-1+1+5+z}{4}\right]$

but $\left[\frac{13+x}{4}, \frac{5+y}{4}, \frac{5+z}{4}\right] = (4, 2, 2)$ [given]

$\Rightarrow \frac{13+x}{4} = 4 \Rightarrow 13 + x = 16 \Rightarrow x = 3,$

$\Rightarrow \frac{5+y}{4} = 2 \Rightarrow 5 + y = 8 \Rightarrow y = 3,$

$\Rightarrow \frac{5+z}{4} = 2 \Rightarrow 5 + z = 8 \Rightarrow z = 3$

∴ Fourth vertex D(3, 3, 3)

12. Fin the fourth vertex of the parallelogram whose consecutive vertices are (2, 4, -1), (3, 6, -1) and (4, 5, 1).

Sol: (2, 4, -1), (3, 6, -1) and (4, 5, 1) are the consecutive vertices of a parallelogram,

then the fourth vertex of the parallelogram

$= (x_1 + x_3 - x_2, y_1 + y_3 - y_2, z_1 + z_3 - z_2)$

= (2 + 4 – 3, 4 + 5 - 6, -1 + 1 +1) = (3, 3, 1)

MORE QUESTIONS FOR PRACTICE

1. Find the distance between the points (3, 4, -2), (1, 0, 7).
2. Show that the points A(-4, 9, 6), B(-1, 6, 6) and C(0, 7, 10) form a right angled triangle.
3. Show that the points A(3, -2, 4), B(1, 1, 1) and C(-1, 4, -2) are collinear.
4. Show that the points (5, 4, 2), (6, 2, -1) and (8, -2, -7) are collinear.

6. DIRECTIONCOSINES AND DIRECTION RATIOS

- **Direction Cosines of a line:** If a ray makes angles α, β, ϒ with the three coordinate axes then (cos α, cos β, cos ϒ) are called direction cosines (d.c's) denoted by (l, m, n).
- If (l, m, n) are direction cosines of line then $l^2 + m^2 + n^2 = 1$.
- The d.r.'s of the line joining $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ are $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$.
- The d.c's of the line joining $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ are $\pm\left[\frac{x_2-x_1}{2}, \frac{y_2-y_1}{2}, \frac{z_2-z_1}{2}\right]$
- **Direction Ratios of a line**: An ordered triple of numbers which are proportional to the d.c.'s of a line, is called the Direction Ratios(D.r's) of a line.
- If (a, b, c) are direction ratios, (l, m, n) are direction cosines of a lines then $a: b: c = l: m: n$..
- The d.c.'s of the line whose d.r's are (a, b, c) are $\pm\left[\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right]$
- If θ is the acute angle between the lines whose direction ratios are (a_1, b_1, c_1), (a_2, b_2, c_2) then

$$cos\theta = \frac{|a_1a_2+b_1b_2+c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

- Two lines having d.r's (a_1, b_1, c_1), (a_2, b_2, c_2) are

i) Parallel ⟺ $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

ii) Perpendicular ⟺ $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$

LEVEL - 1

1. Find the distance between the midpoint of the line segment $\overline{AB}$ and the point (3, -1, 2) where A= (6,3, -4) and B= (-2, -1, 2).

Sol: Midpoint of AB

$= \left[\frac{6-2}{2}, \frac{3-1}{2}, \frac{-4+2}{2}\right]$ = (2, 1, -1)

∴ Distance between (3, -1, 2) and (2, 1, -1) is

$= \sqrt{(3-2)^2 + (-1-1)^2 + (2+1)^2} = \sqrt{1+4+9}$

$= \sqrt{14}$ Units

2. If the point (1, 2, 3) is changed to the point (2, 3, 1) through translation of axes, find the new origin.

Sol: Let (X, Y, Z) be the new coordinates of (x, y, z)

Given that (x, y, z) = (1, 2, 3) and (X, Y, Z) = (2, 3, 1)

Let (h, k, l) be the new origin

$x = X + h$	$y = Y + k$	$z = Z + l$
$\Rightarrow h = x - X$	$\Rightarrow k = y - Y$	$\Rightarrow l = z - Z$
$\Rightarrow h = 1 - 2$	$\Rightarrow k = 2 - 3$	$\Rightarrow l = 3 - 1$
$\Rightarrow h = -1$	$\Rightarrow k = -1$	$\Rightarrow l = 2$

∴ (-1, 1, 2) is the new origin)

3. A line makes angles 90 ° , 60 ° and 30 ° with the positive direction of x. y, z axes respectively. Find the direction cosines.

Sol: d.c's of the ray are

$(cos90^o, cos60^o, cos30^o) = \left(0, \frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

4. A ray makes angles $\frac{\pi}{3}$, $\frac{\pi}{3}$ with $\overline{OX}$ and $\overline{OY}$ axis respectively. Find the angle made by it with $\overline{OZ}$.

Sol: Let the angle made by the ray with $\overline{OZ}$ be ϒ

$\Rightarrow cos^2\frac{\pi}{3} + cos^2\frac{\pi}{3} + cos^2\gamma = 1$

$\Rightarrow \frac{1}{4} + \frac{1}{4} + cos^2\gamma = 1$

$\Rightarrow cos^2\gamma = 1 - \frac{1}{2}$

$\Rightarrow cos\gamma = \pm\frac{1}{\sqrt{2}}$

$\Rightarrow \gamma = \frac{\pi}{4}$ or $\frac{3\pi}{4}$.

5. If a line makes angles of α, β, γ with the positive direction of x, y, z axes. What is the value of $\sin^2 \alpha + \sin^2 \beta + \sin^2 \Upsilon$?

Sol: If a line makes angle of α, β, ϒ with the positive direction of x, y, z axes then

$\cos^2 \alpha + \cos^2 \beta + \cos^2 \Upsilon = 1$

$\Rightarrow 1 - \sin^2 \alpha + 1 - \sin^2 \beta + 1 - \sin^2 \Upsilon = 1$

$\Rightarrow 3 - (\sin^2 \alpha + \sin^2 \beta + \sin^2 \Upsilon) = 1$

$\Rightarrow \sin^2 \alpha + \sin^2 \beta + \sin^2 \Upsilon = 2$

6. If P($\sqrt{3}$, 1, 2 $\sqrt{3}$) is a point in space find direction cosines of $\overline{OP}$.

Sol: d.r's of $\overline{OP}$ are $(\sqrt{3},\ 1,\ 2\sqrt{3})$

∴ d.c's are $\overline{OP}$ are

$$\pm\left[\frac{\sqrt{3}}{\sqrt{(\sqrt{3})^2+1^2+(2\sqrt{3})^2}}, \frac{1}{\sqrt{(\sqrt{3})^2+1^2+(2\sqrt{3})^2}}, \frac{2\sqrt{3}}{\sqrt{(\sqrt{3})^2+1^2+(2\sqrt{3})^2}}\right]$$

$$= \pm\left[\frac{\sqrt{3}}{\sqrt{3+1+12}}, \frac{1}{\sqrt{3+1+12}}, \frac{2\sqrt{3}}{\sqrt{3+1+12}}\right]$$

$$= \pm\left[\frac{\sqrt{3}}{\sqrt{16}}, \frac{1}{\sqrt{16}}, \frac{2\sqrt{3}}{\sqrt{16}}\right] = \pm\left[\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{2\sqrt{3}}{4}\right]$$

$$= \pm\left[\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}\right]$$

7. Find the d.c.'s of the line joining points (4, -7, 3) and (6, -5, 2)

Sol: Let A(4, -7, 3) , B (6, -5, 2)

d.r's of $\overline{AB}$ are

(6-4, -5+7, 2-3) = (2, 2, -1)

∴ d.c.'s of $\overline{AB}$ are

$$= \left[\frac{2}{\sqrt{4+4+1}}, \frac{1}{\sqrt{4+4+1}}, \frac{\sqrt{3}}{\sqrt{4+4+1}}\right] = \left[\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}\right]$$

8. Find the direction cosines of the line joining the points (-4, 1, 7) , (2, -3, 2).

Sol: Let A(-4, 1, 7) , B(2, -3, 2)

d.r's of $\overline{AB}$ are

= (6-4, -5+7, 2-3) = (2, 2, -1)

∴ d.c.'s of $\overline{AB}$ are (2 + 4, -3-1, 2 – 7)

= (6, -4, -5) = (-6, 4, 5)

$$= \left[\frac{-6}{\sqrt{36+16+25}}, \frac{4}{\sqrt{36+16+25}}, \frac{5}{\sqrt{36+16+25}}\right]$$

$$= \left[\frac{-6}{\sqrt{77}}, \frac{4}{\sqrt{77}}, \frac{5}{\sqrt{77}}\right]$$

9. Find the d.c.'s of a line that makes equal angles with the axes.

Sol: Let a be the angle made by a line with x – axis then d.c.'s of the line are (cos a, cos a, cos a)

But $\cos^2 a + \cos^2 a + \cos^2 a = 1$

$\Rightarrow 3\cos^2 a = 1 \quad \Rightarrow \cos^2 a = \frac{1}{3} \quad \Rightarrow \cos a = \pm \frac{1}{\sqrt{3}}$

$\therefore$ d.c.'s of the line are $\pm\left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{5}{\sqrt{3}}\right]$

10. If the d.c.'s of a line are $\left[\frac{1}{c}, \frac{1}{c}, \frac{1}{c}\right]$ then find 'c'.

Sol: If (l, m, n) are the d.c.'s of a line then

$l^2 + m^2 + n^2 = 1 \quad \Rightarrow \frac{1}{c^2} + \frac{1}{c^2} + \frac{1}{c^2} = 1$

$\Rightarrow \frac{3}{c^2} = 1 \quad \Rightarrow c^2 = 3 \quad \Rightarrow c = \pm\sqrt{3}$

11. Find the angle between the lines whose direction ratios are (1, 1, 2) , ($\sqrt{3}$, $\sqrt{3}$, 0)

Sol: If $'\theta'$ be the angle between the given lines

$$cos\theta = \frac{|1(\sqrt{3}) + 1(-\sqrt{3}) + 2(0)|}{\sqrt{1^2+1^2+2^2}\ \sqrt{(\sqrt{3})^2 + (-\sqrt{3})^2 + 0^2}}$$

$$= \frac{|1\sqrt{3} - \sqrt{3} + 0|}{\sqrt{6}\,\sqrt{6}} \Rightarrow cos\theta = 0$$

$\Rightarrow \theta = 90^o$

12. O is the origin, P (2, 3, 4) and Q (1, k, 1) are points such that $\overline{OP} \perp \overline{OQ}$, then find 'k'.

Sol: D.r.'s of $\overline{OQ}$ = (1, k, 1) , Dr of $\overline{OP}$ = (2, 3, 4)

Since OP $\perp$ OQ $\Rightarrow$ 3(1) + 3(k) + 4(1) = 0

$\Rightarrow$ 2 + 3k + 4 = 0 $\Rightarrow$ 3k + 6 = 0

$\Rightarrow$ k = -2

13. Show that the points (4, 7, 8) , (2, 3, 4) , (-1, -2, 1), (1, 2, 5) are vertices of a parallelogram.

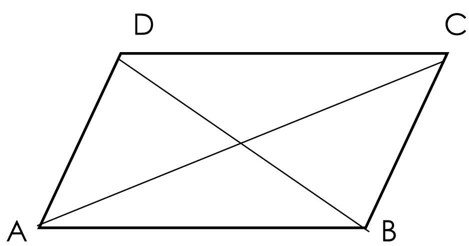
Sol: Let the points are A(4, 7, 8), B(2, 3, 4) , C (-1, -2, 1) , D(1, 2, 5)

Mid point of AC = $\left[\frac{4-1}{2}, \frac{7-2}{2}, \frac{8+1}{2}\right]$

$= \left[\frac{3}{2}, \frac{5}{2}, \frac{9}{2}\right]$

$\therefore$ Mid point of AC = Midpoint of BD

$\Rightarrow$ Given points from a parallelogram.



More Questions for Practice.

14. The direction ratios of a line are (-6, 2, 3). Find its direction cosines.
15. If the d.c.'s of a line are proportional to (1, -2, 1) find its d.c.'s.

LEVEL – 1 (7 Marks)

1. Find angle between the lines whose direction cosines satisfy the equations $l+m+n=0$, $l^2+m^2-n^2=0$

Sol: Given $l+m+n=0 \Rightarrow l=-m-n$ ------- (1)

$l^2+m^2-n^2=0$ -------- (2)

(2) $\Rightarrow (-m-n)^2+m^2-n^2=0 \Rightarrow m^2+n^2+2mn+m^2-n^2=0$

$\Rightarrow 2m^2+2mn=0 \Rightarrow 2m(m+n)=0$

$m=0,\ m+n=0$

Case (i): If $m=0$

(1) $\Rightarrow l=-0-n$

$\Rightarrow l=-n$

$\therefore l:m:n=-n:0:n$

$=-1:0:1$

Case (ii): If $m+n=0 \Rightarrow m=-n$

(1) $\Rightarrow l=-(-n)-n$

$\Rightarrow l=0$

$\therefore l:m:n=0:-n:n$

$=0:-1:1$

$\therefore$ D. r's of the lines are (-1, 0, 1), (0, -1, 1)

If θ is the acute angle between (1) & (2) then

$$cos\theta=\frac{|(-1)(0)+0(-1)+1(1)|}{\sqrt{(-1)^2+0^2+1^2}\sqrt{0^2+(-1)^2+1^2}}$$

$$=\frac{1}{\sqrt{2}.\sqrt{2}}=\frac{1}{2}$$

$cos\theta=\frac{1}{2} \Rightarrow \theta=60^o$

2. Find the angle between the lines whose direction cosines are given by the relations $3l+m+5n=0$, $6mn-2nl+5lm=0$.

Sol: Given $3l+m+5n=0 \Rightarrow m=-3l-5n$ ------- (1)

And $6mn-2nl+5lm=0$ -------------(2)

(2) $\Rightarrow 6n(-3l-5n)-2nl+5l(-3l-5n)=0$

$\Rightarrow -18nl-30n^2-2nl-15l^2-25ln=0$

$\Rightarrow -15l^2-45nl-30n^2=0 \Rightarrow l^2+3ln+2n^2=0$

$\Rightarrow l^2+2ln+ln+2n^2=0 \Rightarrow l(l+2n)+n(l+2n)=0$

$\Rightarrow (l+2n)(l+n)=0$

Case (i) $l+2n=0 \Rightarrow l=-2n$

(1) $\Rightarrow m=-3(-2n)-5n=n$

$\therefore l:m:n=-2n:n:n$

$=-2:1:1$

Case (ii) $l+n=0 \Rightarrow l=-n$

(1) $\Rightarrow m=-3(-n)-5n=-2n$

$\therefore l:m:n=-n:-2n:n$

$=-1:-2:1$

$\therefore$ d.r's of the lines are (-2, 1, 1), (-1, -2, 1)

If θ is the acute angle between (1) & (2) then

$$cos\theta = \frac{|(-2)(-1)+1(-2)+1(1)|}{\sqrt{(-2)^2+1^2+1^2}\sqrt{(-1)^2+(-2)^2+1^2}}$$

$$= \frac{|2-2+1|}{\sqrt{6}.\sqrt{6}} = \frac{1}{6}$$

$$cos\theta = \frac{1}{6} \Rightarrow \theta = cos^{-1}\left[\frac{1}{6}\right]$$

3. Find the direction cosines of two lines which are connected by the relations $l + m + n = 0$ and $mn - 2nl - 2lm = 0$.

Sol: Given $l + m + n = 0 \Rightarrow l = -m - n$ ------- (1)

And $mn - 2nl - 2lm = 0$ -------------(2)

(2) $\Rightarrow mn - 2n(-m-n) - 2m(-m-n) = 0$

$\Rightarrow mn + 2mn + 2n^2 + 2m^2 + 2mn = 0$

$\Rightarrow 2m^2 + 5mn + 2n^2 = 0$

$\Rightarrow 2m^2 + 4mn + mn + 2n^2 = 0 \Rightarrow 2m(m+2n) + n(m+2n) = 0$

$\Rightarrow (m+2n)(2m+n) = 0$

Case (i) $m + 2n = 0 \Rightarrow m = -2n$

(1) $\Rightarrow l = -(-2n) - n = n$

$\therefore l:m:n = n:-2n:n$

$= 1:-2:1$

Case (ii) $2m + n = 0 \Rightarrow n = -2m$

(1) $\Rightarrow l = -m - (-2m) = m$

$\therefore l:m:n = m:m:2m$

$= -1:1:-2$

$\therefore$ d.r's of the lines are (1, -2, 1), (1, 1, -2)

Now, d.c's of the lines are

$$\left[\frac{1}{\sqrt{1^2+(-2)^2+1^2}}, \frac{-2}{\sqrt{1^2+(-2)^2+1^2}}, \frac{1}{\sqrt{1^2+(-2)^2+1^2}}\right] = \left[\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right]$$

and

$$\left[\frac{1}{\sqrt{1^2+1^2+(-2)^2}}, \frac{1}{\sqrt{1^2+1^2+(-2)^2}}, \frac{-2}{\sqrt{1^2+1^2+(-2)^2}}\right] = \left[\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right]$$

4. Find the direction cosines of two lines which are connected by the relations $l - 5m + 3n = 0$ and $7l^2 + 5m^2 - 3n^2 = 0$.

Sol: Given $l - 5m + 3n = 0 \Rightarrow l = 5m - 3n$ ------ (1)

and $7l^2 + 5m^2 - 3n^2 = 0$ -------- (2)

(2) $\Rightarrow 7(5m-3n)^2 + 5m^2 - 3n^2 = 0 \Rightarrow 7(25m^2 + 9n^2 - 30mn) + 5m^2 - 3n^2 = 0$

$\Rightarrow 175m^2 + 63n^2 - 210mn + 5m^2 - 3n^2 = 0$

$\Rightarrow 180m^2 - 210mn + 60n^2 = 0$

$\Rightarrow\ 6m^2 - 7mn + 2n^2 = 0 \Rightarrow 6m^2 - 4mn - 3mn + 2n^2 = 0$

$\Rightarrow 2m(3n - 2n) - n(3m - 2n) = 0 \Rightarrow (3m - 2n)(2m - n) = 0$

Case (i) If $3m - 2n = 0 \Rightarrow m = \frac{2n}{3}$

(1) $\Rightarrow l = 5\left(\frac{2n}{3}\right) - 3n = l = \frac{n}{3}$

$\therefore l: m: n = \frac{n}{3}: \frac{2m}{3}: n$

$= \frac{1}{3}: \frac{2}{3}: 1 = 1: 2: 3$

Case (ii): If $2m - n = 0 \Rightarrow n = 2m$

(1) $\Rightarrow 5m - 3(2m) = -m$

$\therefore l: m: n = -m: m: 2m$

= -1: 1: 2

$\therefore$ d.r's of the lines are (1, 2, 3), (-1, 1, 2)

$\therefore$ d.c's of the lines are

$\left[\frac{1}{\sqrt{1^2+2^2+3^2}}, \frac{2}{\sqrt{1^2+2^2+3^2}}, \frac{3}{\sqrt{1^2+2^2+3^2}}\right] = \left[\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right]$

and

$\left[\frac{-1}{\sqrt{(-1)^2+1^2+2^2}}, \frac{1}{\sqrt{(-1)^2+1^2+2^2}}, \frac{3}{\sqrt{(-1)^2+1^2+2^2}}\right] = \left[\frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right]$

5. Show that the lines whose d.c's are given by $l + m + n = 0$, $2mn + 2nl - 5lm = 0$ are perpendicular to each other.

Sol: Given $l + m + n = 0 \Rightarrow m = -l - n$ ------ (1)

and $2mn + 3nl - 5lm = 0$ -------- (2)

(2) $\Rightarrow 2mn + 3n(-m - n) - 5m(-m - n) = 0 \Rightarrow$

$\Rightarrow 2mn - 3mn - 3n^2 + 5m^2 + 5mn = 0$

$\Rightarrow 5m^2 + 4mn - 3n^2 = 0$

$\Rightarrow 5\left(\frac{m}{n}\right)^2 + 4\left(\frac{m}{n}\right) - 3 = 0$ [$\because$ *divided by 'n'*]

This is a quadratic in $\frac{m}{n}$, this have two roots.

Let those roots are $\frac{m_1}{n_1}, \frac{m_2}{n_2}$

Multiplication of roots $\frac{m_1}{n_1} = \frac{m_2}{n_2} = \frac{-3}{5}$ [$\therefore$ if α, β are the roots of $ax^2 + bx + c = 0$

then $\alpha\beta = \frac{c}{a}$]

$\Rightarrow \frac{m_1 m_2}{n_1 n_2} = \frac{-3}{5} \Rightarrow \frac{m_1 m_2}{3} = \frac{n_1 n_2}{-5}$ ------ (3)

again (2) $\Rightarrow 2n) - l - n) + 3nl - 5l(-l - n) = 0$

$\Rightarrow 2nl - 2n^2 + 3nl + 5l^2 + 5ln = 0$

$\Rightarrow 5l^2 + 6nl - 2n^2 = 0$

$= 5\left[\frac{1}{n}\right]^2 + 6\left[\frac{1}{n}\right] - 2 = 0$ [∵ divided by n^2]

This is a quadratic equation in $\left[\frac{1}{n}\right]$, this have two roots.

Let those roots are $\frac{l_1}{n_1}, \frac{l_2}{n_2}$

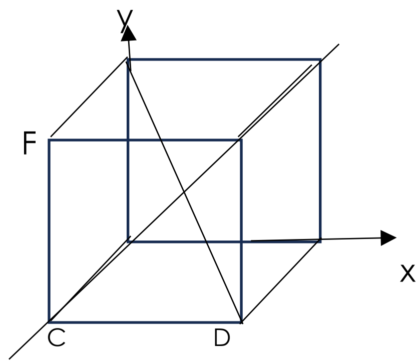
$\Rightarrow \frac{l_1}{n_1} \cdot \frac{l_2}{n_2} = \frac{-2}{5} \Rightarrow \frac{l_1 l_2}{2} = \frac{n_1 n_2}{-5}$ ------ (4)

From (3) & (4) $\frac{l_1 l_2}{2} = \frac{m_1 m_2}{3} = \frac{n_1 n_2}{-5}$

Take $l_1 l_2 + m_1 m_2 + n_1 n_2 = 2 + 3 - 5 = 0$

∴ the lines (1) & (2) are perpendicular

6. Find the angle between the two diagonals of a cube.



Sol: Let OACBDEFG be a cube

Let OA = OB = OC = a

The coordinates of the vertices of a cube are

O(0, 0, 0), A(a, 0, 0), B(0, a, 0), C(0, 0, a)

D(a, 0, a), E(a, a, 0), F(0, a, a), G(a, a, a)

From the figure $\overline{OG}, \overline{AF}, \overline{CE}, \overline{BD}$ are four diagonals.

The D.r's of $\overline{OG}$ are (a – 0, a – 0, a – 0) = (a, a, a)

The D.r's of $\overline{BD}$ are (a – 0, 0 – a , a – 0) = (a, -a, a)

If θ is the angle between $\overline{OG}, \overline{BD}$ then

$cos\theta = \frac{a(a)+a(-a)+(a)}{\sqrt{a^2+a^2+a^2}\sqrt{a^2+(-a)^2+a^2}} = \frac{a^2-a^2+a^2}{\sqrt{3a^2}\sqrt{3a^2}}$

$= \frac{a^2}{3a^2} = \frac{1}{3}$

$\therefore cos\theta = \frac{1}{3} \Rightarrow \theta = cos^{-1}\left[\frac{1}{3}\right]$

∴ Angle between any two diagonals of a cube is $cos^{-1}\left[\frac{1}{3}\right]$

7. If a ray makes angle $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube then find $cos^2\alpha + cos^2\beta + cos^2\gamma + cos^2\delta$.

Sol: : Let OACBDEFG be a cube

Let OA = OB = OC = a

The coordinates of the vertices of a cube are

O(0, 0, 0), A(a, 0, 0), B(0, a, 0), C(0, 0, a)

D(a, 0, a), E(a, a, 0), F(0, a, a), G(a, a, a)

From the figure $\overline{OG}$, $\overline{AF}$,$\overline{CE}$,$\overline{BD}$ are four diagonals.

The D.r's of $\overline{OG}$ are (a – 0, a – 0, a – 0) = (a, a, a)

The D.r's of $\overline{AF}$ are (0 – a, a – 0, a – 0) = (-a, a, a)

The D.r's of $\overline{CE}$ are (a – 0, a – 0, 0 – a) = (a, a, -a)

The D.r's of $\overline{BD}$ are (a – 0, 0 – a , a – 0) = (a, -a, a)

Let (l,m,n) be the d.c's of the given ray, then we know that $l^2+m^2+n^2=1$

The ray makes $\alpha,\beta,\gamma,\delta$ with $\overline{OG}$, $\overline{AF}$,$\overline{CE}$,$\overline{BD}$ respectively then

$$cos\alpha=\frac{la+ma+na}{\sqrt{l^2+m^2+n^2}\sqrt{a^2+a^2+a^2}}=\frac{a(l+m+n)}{\sqrt{1}\sqrt{3a^2}}=\frac{l+m+n}{\sqrt{3}}$$

Similarly, we get $cos\beta=\frac{-l+m+n}{\sqrt{3}}$, $cos\gamma=\frac{l+m-n}{\sqrt{3}}$, $cos\delta=\frac{l-m+n}{\sqrt{3}}$

Now, $cos^2\alpha+cos^2\beta+cos^2\gamma+cos^2\delta=\left[\frac{l+m+n}{\sqrt{3}}\right]^2+\left[\frac{-l+m+n}{\sqrt{3}}\right]^2+\left[\frac{l+m-n}{\sqrt{3}}\right]^2+\left[\frac{l-m+n}{\sqrt{3}}\right]^2$

$=\frac{1}{3}[l^2+m^2+n^2+2lm+2mn+2nl+l^2+m^2+n^2-2lm+2mn-2nl+l^2+m^2+n^2+2lm-2mn-2nl+l^2+m^2+n^2-2lm-2mn+2nl]$

$$=\frac{1}{3}[4(l^1+m^2+n^2)]=\frac{4}{3}$$

$$\therefore cos^2\alpha+cos^2\beta+cos^2\gamma+cos^2\delta=\frac{4}{3}$$

8. The vertices of a triangle are A(1, 4, 2), B(-2, 1, 2), C(2, 3, -4). Find ∠A, ∠B, ∠C.

Sol: Given that A(1, 4, 2), B(-2, 1, 2), C(2, 3, -4)

D.r's of $\overline{AB}$ are (-2-1, 1-4, 2-2) = (-3, -3, 0)

D.r's of $\overline{BC}$ are (2+2, 3-1, -4-2) = (4, 2, -6)

D.r's of $\overline{AC}$ are (2-1, 3-4, 4-2) = (1, -1, 6)

Let A be the angle between $\overline{AB}$ and $\overline{AC}$ then

$$cosA=\frac{|a_1a_2+b_1b_2+c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}=\frac{|(-3)1+(-3)(-1)+0(6)|}{\sqrt{(3)^2+(-3)^2+0^2}\sqrt{1^2+(-1)^2+(-6)^2}}=\frac{|-3+3+0|}{\sqrt{9+9+0}\sqrt{1+1+36}}$$

Hence, $\cos A=0$ ⇒ A = 90°

Let B be the angle between $\overline{AB}$ and $\overline{BC}$ then

$$cosB=\frac{|(-3)4+(-3)2+0(-6)|}{\sqrt{(-3)^2+(-3)^2+0^2}\sqrt{4^2+2^2+(-6)^2}}=\frac{|-12-6+0|}{\sqrt{9+9+0}\sqrt{16+4+36}}$$

$$=\frac{18}{\sqrt{18}\sqrt{56}}=\sqrt{\frac{18}{56}}=\frac{3}{2\sqrt{7}}\Rightarrow B=cos^{-1}\left(\frac{3}{2\sqrt{7}}\right)$$

Let C be the angle between $\overline{BC}$ and $\overline{AC}$ then

$$cosC=\frac{|4(1)+2(-1)+(-6)(-6)|}{\sqrt{4^2+4^2+(-6)^2}\sqrt{1^2+(-1)^2+(-6)^2}}=\frac{|4-2+36|}{\sqrt{16+4+36}\sqrt{1+1+36}}$$

$= \frac{38}{\sqrt{56}\sqrt{38}} = \sqrt{\frac{38}{56}} = \sqrt{\frac{19}{28}} \Rightarrow B = cos^{-1}\left(\sqrt{\frac{19}{28}}\right)$

9. If (6, 10, 10), (1, 0, -5), (6, -10, 0) are vertices of a triangle. Find the direction ratios of its side, determine whether it is right angled or isosceles.

Sol: Let A(6, 10, 10), B(1, 0, -5), C(6, -10, 0)

D. r's of $\overline{AB}$ are (1-6, 0-10, 5-10) = (-5, -10, -15) = (1, 2, 3) ------- (1)

D. r's $\overline{BC}$ are (6-1, -10-0, 0+5) = (5, -10, 5) = (1, -2, 1) ------- (2)

D. r's $\overline{AC}$ are (6-6, -10-10, 0-10) = (0, -20, -10) = (0, 2, 1) ------- (3)

Now from (1) & (2)

(1)(1) + (2)(-2) + (3)(1) = 1 – 4 + 3 = 0

$\Rightarrow \llcorner ABC = 90^o$

∴ ΔABC is right angled

LEVEL – 2 (7 Marks)

1. A(1, 8, 4), B(0, -11, 4), C(2, -3, 1) are three points and 'D' is the foot of the perpendicular from A to BC. Find the coordinates of D.

Sol: Given points are A(1, 8, 4), B(0, -11, 4), C(2, -3, 1)

Suppose D divides $\overline{BC}$ in the ration k : 1

Then D = $\left[\frac{2k}{k+1}, \frac{-3k-11}{k+1}, \frac{k+4}{k+1}\right]$

D. r's of $\overline{AD}$ = $\left[\frac{2k}{k+1} - 1, \frac{-3k-11}{k+1} - 8, \frac{k+4}{k+1} - 4\right]$

= $\left[\frac{k-1}{k+1}, \frac{-11k-19}{k+1}, \frac{-3k}{k+1}\right]$

D.r's of $\overline{BC}$ = (2-0, -3+11, 1-4) = (2, 8, -3)

But $\overline{AD} \perp \overline{BC} \Rightarrow 2\left[\frac{k-1}{k+1}\right] + 8\left[\frac{-11k-19}{k+1}\right] - 3\left[\frac{-3k}{k+1}\right]$

⇒ 2k – 2 – 88k – 152 + 9k = 0 ⇒ 77k + 154 = 0

⇒ 77k = -154 ⇒ k =-2

∴ D = = $\left[\frac{2(-2)}{-2+1}, \frac{-3(-2)-11}{-2+1}, \frac{-2+4}{-2+1}\right]$ = (4, 5, -2)

2. If (l_1, m_1, n_1), (l_2, m_2, n_3) are d.c's of two intersecting lines. Show that d.c's of two lines bisecting the angles between them are proportional to $l_1 \pm l_2$, $m_1 \pm m_2$, $n_1 \pm n_2$.

Sol: Let OA, OB are the given lines

Let A, B are the points at unit distances from O.

⇒ Co-ordinates of A are (l_1, m_1, n_1)

⇒ Co-ordinates B are (l_2, m_2, n_3)

Mid poind of AB = P$\left(\frac{l_1+l_2}{2}, \frac{m_1+m_2}{2}, \frac{n_1+n_2}{2}\right)$

∴ OP bisects ∟AOB ⇒ D.r's of OP are $(l_1 + l_2, m_1 + m_2, n_1 + n_2)$

Suppose 'B' is a point on the line OB as OB = OB' =1

Co-ordinates of 'B' are $(-l_1, -m_2, -n_2)$

Mid point of AB' = Q$\left(\frac{l_1 - l_2}{2}, \frac{m_1 - m_2}{2}, \frac{n_1 - n_2}{2}\right)$

∴ OQ bisects ∟AOB ⇒ D.r's of OQ are $(l_1 - l_2, m_1 - m_2, n_1 - n_2)$

∴ The d.c's of two lines bisecting the angles between them are

proportional to $l_1 \pm l_2,\ m_1 \pm m_2,\ n_1 \pm n_2$

3. A(-1, 2, -3), B(5, 0, 6), C(0, 4, -1) are three points. Show that the direction cosines of the bisectors of ∟BAC are proportional to (25, 8, 5) and (-11, 20, 23).

Sol: D.r's of $\overline{AB}$ are (5+1, 0-2, -6+3) = (6, -2, -3)

D.r's of $\overline{AB}$ are $\left[\frac{6}{\sqrt{36+4+9}}, \frac{-2}{\sqrt{36+4+9}}, \frac{-3}{\sqrt{36+4+9}}\right]$

= $\left[\frac{6}{\sqrt{49}}, \frac{-2}{\sqrt{49}}, \frac{-3}{\sqrt{49}}\right]$ = $\left(\frac{6}{7}, \frac{-2}{7}, \frac{-3}{7}\right)$

D.r's of $\overline{AC}$ are (0+1, 4-2, -1+3) = (1, 2, 2)

D.c's of $\overline{AC}$ are $\left(\frac{1}{\sqrt{1+4+4}}, \frac{2}{\sqrt{1+4+4}}, \frac{2}{\sqrt{1+4+4}}\right)$

= $\left(\frac{1}{\sqrt{9}}, \frac{2}{\sqrt{9}}, \frac{2}{\sqrt{9}},\right)$ = $\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

D.r's of bisectors of ∟BAC are proportional to

$\left(\frac{6}{7} \pm \frac{1}{3},\ \frac{-2}{7} \pm \frac{2}{3}, \frac{-3}{7} \pm \frac{2}{3}\right)$

= $\left[\frac{18 \pm 7}{21}, \frac{-6 \pm 14}{21}, \frac{-9 \pm 14}{21}\right]$

= $\left[\frac{25}{21}, \frac{8}{21}, \frac{5}{21}\right]$ or $\left[\frac{11}{21}, \frac{-20}{21}, \frac{-23}{21}\right]$

= $(25, 8, 5)$ OR $(11, -20, -23)$

CHAPER 7 ; THE PLANE (2M)

KEY POINTS

- Plane: A Plane is a surface such that the line joining two points on the surface lies entirely on it.
- Equation of the plane in general form is $ax + by + cz + d = 0$.
- The d.r's of the normal to the plane $ax + by + cz + d = 0$ are a, b, c.
- Equation of a plane in tntercept form is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.
- Equation any plane parallel to $ax + by + cz + d = 0$ is $ax + by + cz + k = 0$.
- Equation of the plane passing through (x_1, y_1, z_1) and parallel to $ax + by + cz + d = 0$ is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$.
- Equation of the plane passing through (x_1, y_1, z_1) and perpendicular to the ray having d. r's (a, b, c) is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$
- Equation of the plane passing through $A(x_1, y_1, z_1)$, $B(x_1, y_2, z_2)$, $C(x_3, y_3, z_3)$ is
$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$
- Equation of the plane in normal form is $lx + my + nz = P$ where (l, m, n) are the d. c's of the normal to the plane and 'P' is the perpendicular distance to the plane from the origin.
- Perpendicular distance from (0, 0, 0) to the plane $ax + by + cz + d = 0$ is $\frac{|d|}{\sqrt{a^2+b^2+c^2}}$
- Perpendicular distance from $P(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is $\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$.
- Distance between the parallel planes $ax + by + cz + d_1 = 0$, $ax + by + cz + d_2 = 0$ is $\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$.
- If θ is the angle between the planes $ax_1 + by_1 + cz_1 + d_1 = 0$, $ax_2 + by_2 + cz_2 + d_2 = 0$ then $cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$.
- The plane $ax_1 + by_1 + cz_1 + d_1 = 0$, $ax_2 + by_2 + cz_2 + d_2 = 0$ are (i) parallel ⇔ $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{c_1}{c_2}$

  (ii) Perpendicular ⇔ $a_1a_2 + b_1b_2 + c_1c_2 = 0$

LEVEL – 1 (2M)

1. Reduce the equation $x + 2y - 3z - 6 = 0$ of the plane to normal form.

Sol: given equation is $x + 2y - 3z - 6 = 0 \Rightarrow x + 2y - 3z = 6$

Divide with $\sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$

$\Rightarrow \frac{x}{\sqrt{14}} + \frac{2y}{\sqrt{14}} - \frac{3z}{\sqrt{14}} = \frac{6}{\sqrt{14}} \Rightarrow \frac{1}{\sqrt{14}}x + \frac{2}{\sqrt{14}}y - \frac{3}{\sqrt{14}}z = \frac{6}{\sqrt{14}}$

2. Find the d. c's of the normal to the plane $x + 2y + 2z - 4 = 0$.

Sol: D.c's of the normal to the plane $ax + by + cz + d = 0$ are

$\pm\left[\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right]$

$= \pm\left[\frac{1}{\sqrt{1^2+2^2+2^2}}, \frac{2}{\sqrt{1^2+2^2+2^2}}, \frac{2}{\sqrt{1^2+2^2+2^2}}\right] = \pm\left[\frac{1}{\sqrt{9}}, \frac{2}{\sqrt{9}}, \frac{2}{\sqrt{9}}\right]$

$= \pm\left[\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right]$

3. Find the intercepts of the plane $4x + 3y - 2z + 2 = 0$ on the coordinate axes.

Sol: Given Plane equation is $4x + 3y - 2z + 2 = 0$

$\Rightarrow 4x + 3y - 2z = -2 \Rightarrow \frac{4x}{-2} + \frac{3y}{-2} - \frac{2z}{-2} = 1$

$\Rightarrow \frac{x}{-1/2} + \frac{y}{-2/3} + \frac{z}{1} = 1$

$\therefore$ Intersection of the given plane are $\frac{-1}{2}, \frac{-2}{3}, 1$

4. Write the equation of the plane $4x - 4y + 2z + 5 = 0$ into intercept form.

Sol: Given Plane equation is $4x - 4y + 2z + 5 = 0$

$\Rightarrow 4x - 4y + 2z = -5 \Rightarrow \frac{4x}{-5} - \frac{4y}{-5} + \frac{2z}{-5} = \frac{-5}{-5}$

$\Rightarrow \frac{x}{-5/4} + \frac{y}{5/4} + \frac{z}{-5/2} = 1$

This is of the form $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$\therefore\ a = \frac{-5}{4},\ b = \frac{5}{4}, c = \frac{-5}{2}$

5. Find the equation of the plane whose intercepts on x, y, z axes are 1, 2, 4 respectively.

Sol: Formula: Equation of the plane with intercepts a, b, c is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

$\Rightarrow \frac{x}{1}+\frac{y}{2}+\frac{z}{4}=1 \Rightarrow \frac{4x+2y+z}{4}=1$

$\Rightarrow 4x+2y+z=4$

$\Rightarrow 4x+2y+z-4=0$

6. Find the angle between the planes $x+2y+2z-5=0$ and $3x+3y+2z-8=0$.

Sol: given planes are $x+2y+2z-5=0$ --------- (1) and

$3x+3y+2z-8=0$ --------- (2)

If θ is the angle between (1) and (2) then

$$cos\theta=\frac{|1(3)+2(3)+2(2)|}{\sqrt{1^2+2^2+2^2}\sqrt{3^2+3^2+2^2}}=\frac{|3+6+4|}{\sqrt{9}\sqrt{22}}=\frac{13}{3\sqrt{22}}$$

$$\theta=cos^{-1}\left[\frac{13}{3\sqrt{22}}\right]$$

7. Find the angle between the planes $2x-y+z=6$ and $x+y+2z=7$.

Sol: given planes are $2x-y+z=6$ and $x+y+2z=7$

If θ is the angle between the planes then

$$cos\theta=\frac{|2(1)-1(1)+1(2)|}{\sqrt{2^2+(-1)^2+1^2}\sqrt{1^2+1^2+2^2}}=\frac{|2-1+2|}{\sqrt{6}\sqrt{6}}=\frac{3}{6}=\frac{1}{2}$$

$$cos\theta=\frac{1}{2}\Rightarrow \theta=60^o$$

8. Find the equation of the plane passing through (1, 1, 1) and parallel to the plane $x+2y+3z-7=0$.

Sol: equation of plane passing through (x_1,y_1,z_1) and parallel to $ax+by+cz+d=0$ is

$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$

$\Rightarrow 1(x-1)+2(y-1)+3(z-1)=0$

$\Rightarrow x-1+2y-2+3z-3=0$

$\therefore x+2y+3z-6=0$

9. Find the equation of the plane if the foot of the perpendicular from origin to the plane is (2, 3, -5).

Sol: equation of the required plane is ‘

$\Rightarrow 2(x-2)+3(y-3)-5(z+5)=0$

$\Rightarrow 2x-4+3y-9-5z-25=0$

$\therefore 2x+3y-5z-38=0$

10. Find the equation of the plane through (-1, 6, 2) and perpendicular to the join of (1, 2, 3) and (-2, 3, 4).

Sol: Let A(1, 2, 3), B(-2, 3, 4)

D. r’s pf $\overline{AB}$ are (-2,-1, 3 – 2, 4 – 3) = (-3, 1, 1)

Equation of the plane passing through C(-1, 6, 2) and perpendicular to $\overline{AB}$ is

$= -3(x+1)+1(y-6)+1(z-2)$

$= -3x-3+y-6+z-2=0$

$= -3x+y+z-11=0$

$\therefore 3x-y-z+11=0$

11. Find the equation of the plane passes through (-2, 1, 3) and having (3, -5, 4) as d. r’s of its normal.

Sol: equation of the required plane is

$= 3(x-2)-5(y-1)+4(z-3)=0$

$= 3x-6-5y+5+4z-12=0$

$= 3x-5y+4z-1=0$

12. Find the equation of the plane parallel to the zx-plane and passing through (0, 4, 4)

Sol: equation of the zx-plane is $y=0$

Equation of the parallel to zx-plane is $y=k$ -------- (1)

This passes through (0, 0, 4) $\Rightarrow y=k \Rightarrow 4=k$

∴ Equation of the required plane : $y = 4$

13. Find the equation of the plane passing through (2, 3, 4) and perpendicular to x-axis.

Sol: d. r's of x-axis are (1, 0, 0)

∴ Equation of the required plane is

$= 1(x-1) + 0(y-3) + 0(z-4) = 0$

$\Rightarrow x = 2$

MORE QUESTIONS FOR PRACTICE

1. Find the equation of the plane having intercepts 2, 3, 4.
2. Find the equation of the plane if the foot of the perpendicular from origin to the plane is (1, 3, -5).

8. LIMITS AND CONTINUITY

KEY POINTS

I. Standard Limits

- `$\lim_{x\to a} x^n = a^n$
- $\lim_{x\to a} \frac{x^n-a^n}{x-a} = n.a^{n-1}$
- $\lim_{x\to a} \frac{x^m-a^m}{x^n-a^n} = \frac{m}{n}.a^{m-n}$
- $\lim_{x\to 0} \frac{sinx}{x} = 1$
- $\lim_{x\to 0} \frac{tanx}{x} = 1$
- $\lim_{x\to 0} cosx = 1$
- $\lim_{x\to 0}(1+x)^{1/x} = e$
- $\lim_{x\to \infty}\left[1+\frac{1}{x}\right]^x = e$
- $\lim_{x\to 0} \frac{e^x-1}{x} = 1$
- $\lim_{x\to 0} \frac{a^x-1}{x} = \log a$
- $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$
- $\lim_{x\to \infty} \frac{1}{x} = 0$
- $\lim_{x\to -\infty} \frac{1}{x} = 0$

II. Indeterminate forms:

- $\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty,\ \infty-\infty, 0^o,\ \infty^o,\ 0^\infty$

III. Left hand limit [L.H.L]:

- If $f(x)$ approaches to $'l'$ as $'x'$ approaches to $'a'$ from left, then we say that the left hand limit of $f(x)$ is $'l'$.

IV. Right Hand Limit [R.H.L]

- If $f(x)$ approaches to $'l'$ as $'x'$ approaches to $'a'$ from right, then we say that the right hand limit of $f(x)$ is $'l'$. We write this as $\lim_{x\to a^+} f(x) = 1$

V. Continuity: A function $f(x)$ is said to be continuous if and only is

- $\lim_{x\to a} f(x) = f(a)$ (OR)
- $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$

VI. If the above condition is not satisfied the function f is said to be discontinuous.

2 Marks Questions (LEVEL – 1)

1. Evaluate $\lim_{x \to 2} \frac{x-2}{x^3-8}$

Sol: $\lim_{x \to 2} \frac{x-2}{x^3-8}$

$= \lim_{x \to 2} \frac{x-2}{(x-2)(x^2+2x+48)}$

$= \lim_{x \to 2} \frac{1}{(x^2+2x+48)}$

$= \frac{1}{2^2+2(2)+4} = \frac{1}{12}$

2. Evaluate $\lim_{x \to 3} \frac{x^2-8x+15}{x^2-9}$

Sol: $\lim_{x \to 3} \frac{x^2-8x+15}{x^2-9}$

$= \lim_{x \to 3} \frac{(x-5)(x-3)}{(x+3)(x-3)}$

$= \lim_{x \to 3} \frac{x-5}{x+3} = \frac{3-5}{3+3}$

$= \frac{-2}{6} = \frac{1}{3}$

3. Evaluate $\lim_{x \to 3} \frac{x^3-6x^2+9x}{x^2-9}$

Sol: $\lim_{x \to 3} \frac{x^3-6x^2+9x}{x^2-9}$

$= \lim_{x \to 3} \frac{x(x^2-6x+9)}{(x-3)(x+3)}$

$= \lim_{x \to 3} \frac{x(x-3)^2}{(x-3)(x+3)}$

$= \lim_{x \to 3} \frac{x(x-3)}{(x+3)} = \frac{3(3-3)}{(3+3)}$

$= \frac{3(0)}{6} = 0$

4. Evaluate $\lim_{x \to 2} \left[\frac{1}{x-2} - \frac{4}{x^2-4}\right]$

Sol: $\lim_{x \to 2} \left[\frac{1}{x-2} - \frac{4}{x^2-4}\right]$

$= \lim_{x \to 2} \left[\frac{x+2-4}{x^2-4}\right]$

$= \lim_{x\to 0} \frac{x-2}{(x-2)(x+2)} = \lim_{x\to 0} \frac{1}{(x+2)}$

$= \frac{1}{2+2} = \frac{1}{4}$

5. Evaluate $\lim_{x\to 0} \frac{e^{7x}-1}{x}$

Sol: $\lim_{x\to 0} \frac{e^{7x}-1}{x}$

$= \lim_{7x\to 0} \frac{e^{7x}-1}{7x} \times 7 \qquad \left[\because \lim_{x\to 0} \frac{e^{x}-1}{x} = 1\right]$

$= (1) \times 7$

$= 7$

6. Evaluate $\lim_{x\to 0} \frac{e^{3x}-1}{x}$

Sol: $\lim_{x\to 0} \frac{e^{3x}-1}{x}$

$= \lim_{3x\to 0} \frac{e^{3x}-1}{x} \times 3 \qquad \left[\because \lim_{x\to 0} \frac{e^{x}-1}{x} = 1\right]$

$= 1 \times 3 = 3$

7. Evaluate $\lim_{x\to a} \frac{\tan(x-a)}{x^2-a^2}$

Sol: $\lim_{x\to a} \frac{\tan(x-a)}{x^2-a^2}$

$= \lim_{x\to a} \frac{\tan(x-a)}{(x-a)(x+a)}$

$= \lim_{x\to a} \frac{\tan(x-a)}{(x-a)} . \frac{1}{x+a}$

$= \lim_{(x-a)\to 0} \frac{\tan(x-a)}{(x-a)} . \lim_{x\to a} \frac{1}{x+a}$

$= 1 . \frac{1}{a+a} = \frac{1}{2a}$

8. Evaluate $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$

Sol: $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$

$= \lim_{x\to 0} \frac{(\sqrt{x+1}-1)(\sqrt{1+x}+1)}{x(\sqrt{1+x}+1)}$

$$= \lim_{x\to 0} \frac{1+x-1}{x(\sqrt{1+x}+1)}$$

$$= \lim_{x\to 0} \frac{x}{x(\sqrt{1+x}+1)} = \lim_{x\to 0} \frac{1}{(\sqrt{1+x}+1)}$$

$$= \frac{1}{\sqrt{1+0}+1} = \frac{1}{1+1} = \frac{1}{2}$$

9. $\lim_{x\to\infty} \sqrt{x+1} - \sqrt{x}$

Sol: $\lim_{x\to\infty} \sqrt{x+1} - \sqrt{x}$

$$= \lim_{x\to\infty} \frac{(\sqrt{x+1}-\sqrt{x})(\sqrt{x+1}+x)}{(\sqrt{x+1}+\sqrt{x})}$$

$$= \lim_{x\to\infty} \frac{x+1-x}{(\sqrt{x+1}+\sqrt{x})}$$

$$= \lim_{x\to\infty} \frac{1}{(\sqrt{x+1}+\sqrt{x})} = \lim_{x\to\infty} \frac{\frac{1}{\sqrt{x}}}{\sqrt{\frac{x}{x}+\frac{1}{x}}+\sqrt{\frac{x}{x}}}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{\sqrt{x}}}{\sqrt{1+\frac{1}{x}+1}}$$

$$= \frac{0}{\sqrt{1+0+1}} = \frac{0}{2} = 0$$

10. Evaluate $\lim_{x\to\infty} \sqrt{x^2+x} - x$

Sol: $\lim_{x\to\infty} \sqrt{x^2+x} - x$

$$= \lim_{x\to\infty} \frac{(\sqrt{x^2+x}-x)(\sqrt{x^2+x}+x)}{\sqrt{x^2+x}+x}$$

$$= \lim_{x\to\infty} \frac{x^2+x-x^2}{\sqrt{x^2+x}+x}$$

$$= \lim_{x\to\infty} \frac{x}{\sqrt{x^2+x}+x}$$

$$= \lim_{x\to\infty} \frac{\frac{x}{x}}{\sqrt{\frac{x^2+x}{x^2}}+\frac{x}{x}} = \lim_{\frac{1}{x}\to 0} \frac{1}{\sqrt{1+\frac{1}{x}+1}}$$

$$= \frac{1}{\sqrt{1+0}+1} = \frac{1}{1+1} \qquad \left[\because x \to \infty = \frac{1}{x} \to 0\right]$$

$$= \frac{1}{2}$$

11. Evaluate $\lim_{x\to 0}\frac{3^x-1}{\sqrt{1+x}-1}$

Sol: $\lim_{x\to 0}\frac{3^x-1}{\sqrt{1+x}-1}$

$= \lim_{x\to 0}\left[\frac{3^x-1}{x}\right]\left[\frac{x}{\sqrt{1+x}-1}\right]$

$= \lim_{x\to 0}\left[\frac{3^x-1}{x}\right].\lim_{x\to 0}\left[\frac{x}{\sqrt{1+x}-1}\right]$

$= \log 3.\lim_{x\to 0}\left[\frac{x}{\sqrt{1+x}-1}\right]$

$= \log 3.\lim_{x\to 0}\left[\frac{x(\sqrt{1+x}+1)}{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}\right]$

$= \log 3.\lim_{x\to 0}\left[\frac{x(\sqrt{1+x}+1)}{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}\right]$

$= \log 3.\lim_{x\to 0}\left[\frac{x(\sqrt{1+x}+1)}{1+x-1}\right]$

$= \log 3.\lim_{x\to 0}\left[\left(\sqrt{1+x}+1\right)\right]$

$= \log 3.\ \left(\sqrt{1+0}+1\right)$

$= \log 3\ (1+1)$

$= 2\log 3$

12. $\lim_{x\to 0}\frac{e^x-1}{(\sqrt{1+x}-1)}$

Sol: $\lim_{x\to 0}\frac{e^x-1}{\sqrt{1+x}-1}$

$= \lim_{x\to 0}\left[\frac{e^x-1}{x}\right]\left[\frac{x}{\sqrt{1+x}-1}\right]$

$= \lim_{x\to 0}\left[\frac{e^x-1}{x}\right].\lim_{x\to 0}\left[\frac{x}{\sqrt{1+x}-1}\right]$

$= 1.\lim_{x\to 0}\left[\frac{x}{\sqrt{1+x}-1}\right]$

$= \lim_{x\to 0}\left[\frac{x(\sqrt{1+x}+1)}{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}\right]$

$= \lim_{x\to 0}\left[\frac{x(\sqrt{1+x}+1)}{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}\right]$

$= \lim_{x\to 0}\left[\frac{x(\sqrt{1+x}+1)}{1+x-1}\right]$

$= \log 3. \lim_{x\to 0}\left[\left(\sqrt{1+x}+1\right)\right]$

$= \left(\sqrt{1+0}+1\right)$

$= (1+1)$

$= 2$

13. Find $\lim_{x\to 0}\frac{e^{3+x}-e^3}{x}$

Sol: $\lim_{x\to 0}\frac{e^{3+x}-e^3}{x}$

$= \lim_{x\to 0}\frac{e^3\ (e^x-1)}{x}$

$= e^3 \lim_{x\to 0}\frac{(e^x-1)}{x}$

$= e^3\ (1) = e^3$

14. Find $\lim_{x\to 1}\frac{\sin(x-1)}{x^2-1}$

Sol: $\lim_{x\to 1}\frac{\sin(x-1)}{x^2-1}$

$= \lim_{x\to 1}\frac{\sin(x-1)}{(x-1)(x+1)}$

$= \lim_{x\to 1}\frac{\sin(x-1)}{(x-1)} \times \frac{1}{(x+1)}$

$= \lim_{x\to 1}\frac{\sin(x-1)}{(x-1)} . \lim_{x\to 1}\frac{1}{(x+1)}$

$= 1. \frac{1}{1+1} = \frac{1}{2}$

15. $\lim_{x\to 0}\frac{\sin \mathrm{ax}}{\sin bx}$

Sol: $\lim_{x\to 0}\frac{\sin ax}{\sin bx}$

$= \lim_{x\to 0}\frac{\frac{\sin ax}{x}}{\frac{\sin bx}{x}}$

$= \frac{\lim_{x\to 0}\frac{\sin ax}{x}}{\lim_{x\to 0}\frac{\sin bx}{x}}$

$= \frac{a}{b}$ $\left[\because \lim_{x\to 0}\frac{sinkx}{x} = k\right]$

16. $\lim_{x\to 0}\frac{\mathrm{sinax}}{x\ cosx}$

Sol: $\lim_{x\to 0}\frac{sinax}{x\ cosx}$

$= \lim_{x\to 0}\frac{sinax}{x} \times \frac{1}{cosx}$

$=a\frac{1}{(\cos 0)}$ $\left[\because \lim_{x\to 0}\frac{sinkx}{x}=k\right]$

$= a\times\frac{1}{1} = a$

17. $\lim_{x\to\pi/2}\frac{cosx}{x-\frac{\pi}{2}}$

Sol: $\lim_{x\to\pi/2}\frac{cosx}{x-\frac{\pi}{2}}$

Put $x-\frac{\pi}{2}=y$ then $x=\frac{\pi}{2}+y$

and $x\to\frac{\pi}{2}\Rightarrow y\to 0$

$\therefore \lim_{x\to\pi/2}\frac{cosx}{x-\frac{\pi}{2}} = \lim_{y\to 0}\frac{cos\left(\frac{\pi}{2}+y\right)}{y}$

$= \lim_{y\to 0}\frac{-siny}{y} = -1$

18. $\lim_{x\to 0}\frac{e^x-sinx-1}{x}$

Sol: $\lim_{x\to 0}\frac{e^x-sinx-1}{x}$

$= \lim_{x\to 0}\left[\frac{e^x-1}{x}-\frac{sinx}{x}\right]$

$= \lim_{x\to 0}\frac{e^x-1}{x} - \lim_{x\to 0}\frac{sinx}{x}$

$= 1 - 1 = 0$

19. $\lim_{x\to 0}\frac{a^x-1}{b^x-1}$

Sol: $\lim_{x\to 0}\frac{a^x-1}{b^x-1}$

$= \lim_{x\to 0}\frac{\left[\frac{a^x-1}{x}\right]}{\left[\frac{b^x-1}{x}\right]}$

$= \frac{\lim_{x\to 0}\left[\frac{a^x-1}{x}\right]}{\lim_{x\to 0}\left[\frac{b^x-1}{x}\right]}$

$= \frac{log_e a}{log_e b}$

20. $\lim_{x\to 0}\frac{1-cos2\,mx}{sin^2mx}$

Sol: $\lim_{x\to 0}\frac{1-cos2mx}{sin^2mx}$

$= \lim_{x\to 0}\frac{2sin^2mx}{sin^2mx}$

$= 2\lim_{x\to 0}\left[\frac{sinmx}{mx}\right]^2\left[\frac{nx}{\sin nx}\right]^2.\frac{m^2x^2}{n^2x^2}$

$= 2(1)^2(1)^2\frac{m^2}{n^2} = \frac{2m^2}{n^2}$

21. $\lim_{x \to 0} \frac{1-cosmx}{1-cosnx}$

Sol: $\lim_{x \to 0} \frac{1-cosmx}{1-cosnx}$

$= \lim_{x \to 0} \frac{2sin^2\frac{mx}{2}}{2sin^2\frac{nx}{2}}$

$= \lim_{x \to 0} \frac{\left(2sin^2\frac{mx}{2}\right)}{\left(\frac{mx}{2}\right)^2} \left(\frac{mx}{2}\right)^2 . \frac{\left(\frac{nx}{2}\right)^2}{\left(2sin^2\frac{nx}{2}\right)} \left(\frac{nx}{2}\right)^2$

$= \lim_{\frac{mx}{2} \to 0} \left(\frac{sin\frac{mx}{2}}{\frac{mx}{2}}\right)^2 \lim_{\frac{nx}{2} \to 0} \left(\frac{sin\frac{nx}{2}}{\frac{nx}{2}}\right)^2 \left(\frac{m^2x^2}{4}\right)\left(\frac{4}{n^2x^2}\right)$

$= 1.1.\frac{m^2}{n^2}$

$=\frac{m^2}{n^2}$

22. $\lim_{x \to \infty} \frac{11x^3-3x+4}{13x^3-5x^2-7}$

Sol: $\lim_{x \to \infty} \frac{11x^3-3x+4}{13x^3-5x^2-7}$

Taking x^3, the highest power of x as common factor in the numerator and denominator.

$= \lim_{x \to \infty} \frac{x^3\left[11-\frac{3}{x^2}+\frac{4}{x^3}\right]}{x^3\left[13-\frac{5}{x}-\frac{7}{x^3}\right]}$

$= \lim_{\frac{1}{x} \to 0} \frac{11-\frac{3}{x^2}+\frac{4}{x^3}}{13-\frac{5}{x}-\frac{7}{x^3}}$

$= \frac{11-0+0}{13-0-0} = \frac{11}{13}$

23. $\lim_{x \to \infty} \frac{x^2+5x+2}{2x^2-5x+1}$

Sol: $\lim_{x \to \infty} \frac{x^2+5x+2}{2x^2-5x+1}$

Dividing the numerator and denominator by x^2, the highest power of x, have

$= \lim_{x \to \infty} \frac{\frac{x^2}{x^2}+\frac{5x}{x^2}+\frac{2}{x^2}}{\frac{2x^2}{x^2}-\frac{5x}{x^2}+\frac{1}{x^2}}$

$= \lim_{x \to 0} \frac{1+\frac{5}{x}+\frac{2}{x^2}}{2-\frac{5}{x}+\frac{1}{x^2}}$

$= \lim_{\frac{1}{x} \to 0} \frac{1+\frac{5}{x}+\frac{2}{x^2}}{2-\frac{5}{x}+\frac{1}{x^2}}$

$= \frac{1+0+0}{2-0+0} = \frac{1}{2}$

24. $\lim_{x\to\infty} \frac{8|x|+3x}{3|x|-2x}$

Sol: If $x \to \infty$ then $x > 0$, hence $|x| = x$

$\therefore \lim_{x\to\infty} \frac{8|x|+3x}{3|x|-2x} = \lim_{x\to\infty} \frac{8x+3x}{3x-2x}$

$= \lim_{x\to\infty} \frac{11x}{x}$

$= \lim_{x\to\infty} 11 = 11$

25. Show that $\lim_{x\to 2^-} \frac{|x-2|}{x-2} = -1$

Sol: When $x \to 2^-$ then $x < 0 \Rightarrow (x-2) < 0 \Rightarrow |x-2| = -(x-2)$

$= \lim_{x\to 2^-} \frac{|x-2|}{x-2} = \lim_{x\to 2^-} \frac{-(x-2)}{x-2} = -1$

26. Show that $\lim_{x\to 0^+} \frac{2|x|}{x} + x + 1 = 3$

Sol: when $x \to 0^+$ the $x > 0$

$= \lim_{x\to 0^+} \frac{2|x|}{x} + x + 1$

$= \lim_{x\to 0^+} \frac{2x}{x} + x + 1 \qquad [\therefore\ |x| = x]$

= 2 + 0 + 1 = 3

27. Evaluate $\lim_{x\to 1} \frac{log_e x}{x-1}$

Sol: Put $y = x - 1$, then $x \to 1 \Rightarrow x - 1 \to 0 \Rightarrow y \to 0$

$\lim_{y\to 0} \frac{lo(1+y)}{y} = 1$

28. Compute $\lim_{x\to 2^+} ([x] + x)$ and $\lim_{x\to 2^-} ([x] + x)$

Sol: When $x \to 2^+$ then $[x] = 2$

Now RHL = $\lim_{x\to 2^+} ([x] + x)$

= 2 + 2 = 4

When $x \to 2^-$ then $[x] = 1$

Now LHL = $\lim_{x\to 2^-} ([x] + x)$

= 1 + 2 = 3

29. Evaluate $\lim_{x\to 2} \frac{2x^2-7x-4}{(2x-1)(\sqrt{x}-2)}$

Sol: $\lim_{x\to 2} \frac{2x^2-7x-4}{(2x-1)(\sqrt{x}-2)}$

$= \frac{2(2)^2-7(2)-4}{(2(2)-1)(\sqrt{2}-2)}$

$= \frac{10}{3(2-\sqrt{2})}$

$= \frac{10(2+\sqrt{2})}{3(2-\sqrt{2})(2+\sqrt{2})}$

$= \frac{10(2+\sqrt{2})}{3(4-2)}$

$= \frac{10(2+\sqrt{2})}{3(2)} = \frac{5(2+\sqrt{2})}{3}$

30. Evaluate $\lim_{x\to 1}\left[\frac{x-1}{x^2-x} - \frac{1}{x^3-3x^2+x}\right]$

Sol: $\lim_{x\to 1}\left[\frac{x-1}{x^2-x} - \frac{1}{x^3-3x^2+x}\right]$

$= \lim_{x\to 1}\left[\frac{x-1}{x(x-1)} - \frac{1}{x^3-3x^2+x}\right]$

$= \lim_{x\to 1}\left[\frac{1}{x} - \frac{1}{x^3-3x^2+x}\right]$

$= \frac{1}{1} - \frac{1}{1-3(1)+1} = \frac{1}{1} - \frac{1}{1}$

$= 1 + 1 = 2$

31. Evaluate $\lim_{x\to 0}\frac{\sqrt[3]{1+x}-\sqrt[3]{1-x}}{x}$

Sol: $\lim_{x\to 0}\frac{(1+x)^{1/3}-(1-x)^{1/3}}{x}$

$= \lim_{x\to 0}\left[\frac{(1+x)^{1/3}-1}{x} - \frac{(1-x)^{1/3}-1}{x}\right]$

$= \lim_{(1+x)\to 1}\left[\frac{(1+x)^{1/3}-1}{(1+x)-1} + \lim_{(1-x)\to 1}\frac{(1-x)^{1/3}-1}{(1-x)-1}\right]$

$= \frac{1}{3}(1)^{\frac{1}{3}-1} + \frac{1}{3}(1)^{\frac{1}{3}-1}$

$= \frac{1}{3} + \frac{1}{3} = \frac{1+1}{3} = \frac{2}{3}$

32. Evaluate $\lim_{x\to 0}\frac{(1+x)^{1/8}-(1-x)^{1/8}}{x}$

Sol: $\lim_{x\to 0}\frac{(1+x)^{1/8}-(1-x)^{1/8}}{x}$

$= \lim_{x\to 0}\left[\frac{(1+x)^{1/8}-1}{x} - \frac{(1-x)^{1/8}-1}{x}\right]$

$= \lim_{(1+x)\to 1}\left[\frac{(1+x)^{1/8}-1}{(1+x)-1} + \lim_{(1-x)\to 1}\frac{(1-x)^{1/8}-1}{(1-x)-1}\right]$

$= \frac{1}{8}(1)^{\frac{1}{8}-1} + \frac{1}{8}(1)^{\frac{1}{8}-1}$

$= \frac{1}{8} + \frac{1}{8} = \frac{1+1}{8}$

$= \frac{2}{8} = \frac{1}{4}$

33. Evaluate $\lim_{x \to a} \frac{\tan(x-a)}{x^2-a^2}$

Sol: $\lim_{x \to a} \frac{\tan(x-a)}{x^2-a^2}$

$= \lim_{x \to a} \frac{\tan(x-a)}{(x-a)(x+a)}$

$= \lim_{x \to a} \frac{\tan(x-a)}{(x-a)} \cdot \frac{1}{(x+a)}$

$= \lim_{(x-a) \to 0} \frac{\tan(x-a)}{(x-a)} \cdot \lim_{(x-a) \to 0} \frac{1}{(x+a)}$

$= 1. \frac{1}{a+a} = \frac{1}{2a}$

34. Find $\lim_{x \to a} \left[\frac{\sin(x-a) tan^2(x-a)}{(x^2-a^2)^2}\right]$

Sol: $\lim_{x \to a} \left[\frac{\sin(x-a) tan^2(x-a)}{(x^2-a^2)^2}\right]$

$= \lim_{x \to a}(x-9) \left[\frac{\sin(x-a)}{x-a}\right]. \lim_{x \to a} \left[\frac{\tan(x-a)}{(x-a)}\right]^2 . \frac{1}{(x+a)^2}$

$= 0.1.1^2 . \frac{1}{2a^2} = 0$

35. Evaluate $\lim_{x \to a} \frac{e^x-e^3}{x-3}$

Sol: $\lim_{x \to a} \frac{e^x-e^3}{x-3}$

$= \lim_{x \to 3} \frac{e^3(e^{x-3}-1)}{x-3}$

$= e^3 \lim_{(x-3) \to 0} \frac{e^{(x-3)}-1}{x-3}$

$= e^3(1) = e^3$

36. Evaluate $\lim_{x \to 0} \frac{e^{sinx}-1}{x}$

Sol: $\lim_{x \to 0} \frac{e^{sinx}-1}{x}$

$= \lim_{x \to 0} \frac{e^{sinx}-1}{sinx}. \left[\frac{sinx}{x}\right]$

$= \lim_{sinx \to 0} \frac{e^{sinx}-1}{sinx}. \lim_{x \to 0} \frac{sinx}{x}$

$= 1.1 = 1$

37. Evaluate $\lim_{x \to 0} \frac{\log(1+5x)}{x}$

Sol: $\lim_{x \to 0} \frac{\log(1+5x)}{x}$

$= \lim_{x \to 0} \frac{\log(1+5x)}{5x} \times 5$

$= 1 \times 5 = 5$

38. Compute $\lim_{x\to\infty} e^{-x^2}$

Sol: $\lim_{x\to\infty} e^{-x^2}$

$= \lim_{x\to\infty} e^{-x^2}$

$= \lim_{x\to\infty} \frac{1}{e^{x^2}}$ $\left[\therefore \lim_{x\to\infty} \frac{1}{x} = 0\right]$

$= 0$

39. Show that $\lim_{x\to 3} \frac{x-3}{\sqrt{|x^2-9|}} = 0$

Sol: $\lim_{x\to 3} \frac{x-3}{\sqrt{|x^2-9|}}$

$= \lim_{x\to 3} \frac{x-3}{\sqrt{|(x-3)(x+3)|}}$

$= \lim_{x\to 3} \frac{\sqrt{|x-3|^2}}{\sqrt{|x-3||x+3|}}$

$= \lim_{x\to 3} \sqrt{\left|\frac{x-3}{x+3}\right|}$

$= \sqrt{\frac{0}{6}} = \frac{0}{\sqrt{6}} = 0$

LEVEL – 2 (2 Marks)

1. Evaluate $\lim_{x\to a} \frac{x\,sina - a\,sinx}{x-a}$

Sol: $\lim_{x\to a} \frac{x\,sina - a\,sinx}{x-a}$

$= \lim_{x\to a} \frac{(x\,sina - a\,sina) - (a\,sinx - a\,sina)}{x-a}$ $(\because subtract\ and\ add\ a\ sina)$

$= \lim_{x\to a} \frac{sina(x-a)}{x-a} - a \lim_{x\to a} \left(\frac{sinx - sina}{x-a}\right)$

$= \lim_{x\to a} sina - a \lim_{x\to a} \frac{2cos\left(\frac{x+a}{2}\right)sin\left(\frac{x-a}{2}\right)}{x-a}$

$= sina - 2a \lim_{x\to a} cos\left(\frac{x+a}{2}\right) . \lim_{x\to a} \frac{sin\left[\frac{x-a}{2}\right]}{x-a}$

$= sina - 2acos\left(\frac{a+a}{2}\right) . \frac{1}{2}$

$= sina - acosa$

2. Evaluate $\lim_{x\to 0} \frac{cosax - cosbx}{x^2}$

Sol: $\lim_{x\to 0} \frac{cosax - cosbx}{x^2}$

$= \lim_{x\to 0} \frac{2\,sin\left(\frac{ax+bx}{2}\right)sin\left(\frac{bx-ax}{2}\right)}{x^2}$ $\left(\because cosC - cosD = 2sin\frac{C+D}{2}\ sin\frac{D-C}{2}\right)$

$= 2\lim_{x\to 0} \left[\frac{sin\left(\frac{a+b}{2}\right)x}{x}\right]\left[\frac{sin\left(\frac{b-a}{2}\right)x}{x}\right]$

$=2\left[\lim_{x\to 0}\frac{sin\left(\frac{a+b}{2}\right)x}{x}\right]\left[\lim_{x\to 0}\frac{sin\left(\frac{b-a}{2}\right)x}{x}\right]$

$= 2\left(\frac{a+b}{2}\right)\left(\frac{b-a}{2}\right)$ $\left[\because \lim_{x\to 0}\frac{sinkx}{x} = k\right]$

$= \frac{b^2-a^2}{2}$

3. If $f(x) = -\sqrt{25-x^2}$ then find $\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}$

Sol: $\lim_{x\to 1}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1}\frac{-\sqrt{25-x^2}+\sqrt{25-1}}{x-1}$

$= \lim_{x\to 1}\frac{\sqrt{24}-\sqrt{25-x^2}}{x-1}$

$= \lim_{x\to 1}\frac{(\sqrt{24}-\sqrt{25-x^2})(\sqrt{24}+\sqrt{25-x^2})}{x-1(\sqrt{24}+\sqrt{25-x^2})}$

$= \lim_{x\to 1}\frac{24-(25-x^2)}{x-1(\sqrt{24}+\sqrt{25-x^2})}$

$= \lim_{x\to 1}\frac{x^2-1}{x-1(\sqrt{24}+\sqrt{25-x^2})}$

$= \lim_{x\to 1}\frac{(x-1)(x+1)}{x-1(\sqrt{24}+\sqrt{25-x^2})}$

$= \lim_{x\to 1}\frac{(x+1)}{(\sqrt{24}+\sqrt{25-x^2})}$

$= \frac{1+1}{\sqrt{24}+\sqrt{25-1}} = \frac{2}{2\sqrt{24}} = \frac{1}{\sqrt{24}}$

4. Evaluate $\lim_{x\to\infty}\frac{x^2-sinx}{x^2-2}$

Sol: we apply sandwich theorem to evaluate the given limit

We know that $-1 \leq sinx \leq 1$

$\Rightarrow 1 \geq -sinx \geq -1$

$\Rightarrow -1 \leq -sinx \leq 1$

$\Rightarrow (x^2-1) \leq x^2 - sinx \leq (x^2+1)$

$\Rightarrow \frac{x^2-1}{x^2-2} \leq \frac{x^2-sinx}{x^2-2} \leq \frac{x^2+1}{x^2-2}$

Let $l(x) = \frac{x^2-1}{x^2-2}, f(x) = \frac{x^2-sinx}{x^2-2}, u(x) = \frac{x^2+1}{x^2-2}$

Now, $\lim_{x\to\infty} l(x) = \frac{x^2-1}{x^2-2}$

$= \lim_{x\to\infty}\left(\frac{1-\frac{1}{x^2}}{1-\frac{2}{x^2}}\right)$

$= \frac{1-0}{1-0} = 1$

Also, $u(x) = \frac{x^2+1}{x^2-2}$

$= \lim_{x\to\infty} \left(\frac{1+\frac{1}{x^2}}{1-\frac{2}{x^2}}\right)$

$= \frac{1+0}{1-0} = 1$

$\lim_{x\to\infty} l(x) = 1 = \lim_{x\to\infty} u(x)$

∴ from sandwich theorem $\lim_{x\to\infty} f(x) = 1$

5. Evaluate $\lim_{x\to\infty} \frac{2+cos^2x}{x+2007}$

Sol: we know that $0 \le cos^2x \le 1 \Rightarrow 2+0 \le 2+cos^2x \le 2+1$

$\Rightarrow \le 2+cos^2x \le 3$

$\Rightarrow \frac{2}{x+2007} \le \frac{2+cos^2x}{x+2007} \le \frac{3}{x+2007}$

Let $l(x) = \frac{2}{x+2007}, f(x) = \frac{2+cos^2x}{x+2007}, u(x) = \frac{3}{x+2007}$

Now, $\lim_{x\to\infty} l(x) = \frac{2}{x+2007}$

$= \lim_{x\to\infty} \left(\frac{2/x}{1+2007/x}\right)$

$= \frac{0}{1+0} = 0$

Also, $u(x) = \frac{3}{x+2007}$

$= \lim_{x\to\infty} \left(\frac{3/x}{1+\frac{2007}{x}}\right)$

$= \frac{0}{1+0} = 0$

$\lim_{x\to\infty} l(x) = 0 = \lim_{x\to\infty} u(x)$

∴ from sandwich theorem $\lim_{x\to\infty} f(x) = 0$

6. Find $\lim_{x\to a} \left[\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}\right]$

Sol: Here we have to rationalize both the numerator and denominator

$= \lim_{x\to a} \frac{(\sqrt{a+2x}-\sqrt{3x})(\sqrt{a+2x}+\sqrt{3x})}{(\sqrt{a+2x}+\sqrt{3x})} . \frac{(\sqrt{3a+x}+\sqrt{4x})}{(\sqrt{3a+x}-\sqrt{4x})(\sqrt{3a+x}+\sqrt{4x})}$

$= \lim_{x\to a} \frac{(a+2x-3x)}{(\sqrt{a+2x}+\sqrt{3x})} . \frac{(\sqrt{3a+x}+\sqrt{4x})}{3a+x-4x}$

$= \lim_{x\to a} \frac{(a-x)(\sqrt{3a+x}+\sqrt{4x})}{(\sqrt{a+2x}+\sqrt{3x})\ 3\ (a-x)}$

$= \frac{\sqrt{3a+a}+\sqrt{4a}}{(\sqrt{a+2a}+\sqrt{3a})3}$

$= \frac{2\sqrt{a}+2\sqrt{a}}{2(\sqrt{3a})3}$

$= \frac{4\sqrt{a}}{2(\sqrt{3a})3} = \frac{2}{3\sqrt{3}}$

SAQ (4 Marks)

7. Check the continuity of $f(x) = \begin{cases} \frac{1}{2}(x^2-4) & if\ 0 < x < 2 \\ 0 & if\ x = 2 \\ 2 - 8x^{-3} & if\ x = 2 \end{cases}$ at 2.

Sol: (i) Given that f(2) = 0

(ii) L.H.L = $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{1}{2}(x^2-4)$

$= \frac{1}{2}(4-4)$ $\left[\because x \to 2^- \Rightarrow x < 2 \Rightarrow f(x) = \frac{1}{2}(x^2-4)\right]$

= 0

(iii) R.H.L = $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \left(2 - \frac{8}{x^3}\right)$

$= 2 - \frac{8}{8}$ $[\because x \to 2^+ \Rightarrow x > 2 \Rightarrow f(x) = 2 - 8x^{-3}]$

= 2 − 1 = 1

Here L.H.L ≠ R.H.L

$\Rightarrow \lim_{x\to 2} f(x)$ does not exist

So, $f(x)$ is not continuous at 2.

8. If f is given by $f(x) = \begin{cases} k^2x - k & if\ x \geq 1 \\ 2 & if\ x < 1 \end{cases}$ is a continuous function on R, then find K.

Sol: Given $f(x)$ is continuous on R. So $f(x)$ is continuous the split point $x = 1$

(i) When $x < 1$

L.H.L = $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2$

= 2 -------- (1)

(ii) When $x > 1$,

R.H.L = $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} k^2x - k$

$= k^2(1) - k$

$= k^2 - k$ ------ (2)

(iii) from (1) & (2)

L.H.L = R.H.L as $f(x)$ is continuous at $x = 1$

$\Rightarrow k^2 - k = 2$

$\Rightarrow k^2 - k - 2 = 0$

$\Rightarrow (k-2)(k+1) = 0$

⇒ k = 2 or -1

9. Find the real constants a, b so that the function f given by $f(x) = \begin{cases} sinx & if\ x \le 0 \\ x^2 + a & if\ 0 < x < 1 \\ bx + 3 & if\ 1 \le x \le 3 \\ -3 & if\ x > 3 \end{cases}$ is continuous on R

Sol: Given that $f(x)$ is continuous on R.

$\Rightarrow f(x)$ is continuous at split point $x = 0, 3$

(a) $f(x)$ is continuous at $x = 0$

(i) L.H.L = $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} sinx$

= sin 0 = 0

(ii) R.H.L = $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x^2 + a)$

= 0^2 + a = a

But $f(x)$ is continuous at $x = 0$

(b) $f(x)$ is continuous at $x = 3$

∴ L.H.L = $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (bx + 3)$

= 3b + 3

(ii) R.H.L = $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (-3)$

= - 3

But $f(x)$ is continuous at $x = 3$

∴ L.H.L = R. H. L

⇒ 3b + 3 = -3

⇒ 3b = - 6

⇒ b = -2

10. Show that $f(x) = \begin{cases} \frac{cosax - cosbx}{2}, & if\ x \ne 0 \\ \frac{1}{2}(b^2 - a^2) & if x = 0 \end{cases}$ is continuous at 0.

Sol: (i) Given f(0) = $\frac{b^2 - a^2}{2}$ ------- (1)

(ii) $\lim_{x \to 0} \frac{cosax - cosbx}{2}$ = = $\lim_{x \to 0} \frac{2\ sin\left(\frac{ax+bx}{2}\right) sin\left(\frac{bx-ax}{2}\right)}{x^2}$ $\left(\because cosC - cosD = 2sin\frac{C+D}{2}\ sin\frac{D-C}{2}\right)$

$= 2\lim_{x \to 0} \left[\frac{sin\left(\frac{a+b}{2}\right)x}{x}\right] \left[\frac{sin\left(\frac{b-a}{2}\right)x}{x}\right]$

$=2\left[\lim_{x \to 0} \frac{sin\left(\frac{a+b}{2}\right)x}{x}\right] \left[\lim_{x \to 0} \frac{sin\left(\frac{b-a}{2}\right)x}{x}\right]$

$= 2\left(\frac{a+b}{2}\right)\left(\frac{b-a}{2}\right)$ $\left[\because \lim_{x \to 0} \frac{sinkx}{x} = k\right]$

$= \frac{b^2-a^2}{2}$ ----- (2)

(iii) from (1) and (2)

$\lim_{x\to 0} f(x) = f(0)$

So, $f(x)$ is continuous at $x = 0$

11. Check the continuity of the following function given by $f(x) = \begin{cases} \frac{x^2-9}{x^2-2x-3}, & if\ 0 < x < 3\ \ and\ \ x \neq 3 \\ 1.5 & if x = 3 \end{cases}$

at the point 3.

Sol: A function $f(x)$ is continuous $\Leftrightarrow \lim_{x\to a} f(x) = f(a)$

Now, $\lim_{x\to 3} \frac{x^2-9}{x^2-2x-3}$

$= \lim_{x\to 3} \frac{(x+3)(x-3)}{(x-3)(x+1)}$

$= \lim_{x\to 3} \frac{(x+3)}{(x+1)}$

$= \frac{3+3}{3+1} = \frac{6}{4}$

= 1.5

Given that f(3) = 1.5

$\therefore \lim_{x\to 3} f(x) = f(3) = 1.5$ i.e., f is continuous at $x = 3$

9. DIFFERENTIATION

SYNOPSIS POINTS

1. Derivative of $f(x)$ using first principles $f^1(x) = \underset{x\to 0}{lt} \frac{f(x+h)-f(x)}{h}$, $f'(a) = \underset{x\to a}{lt} \frac{f(x)-f(a)}{x-a}$.
2. If u, v are functions of x then $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$; $\frac{du}{dx}\left(\frac{u}{v}\right) = \frac{v\left(\frac{du}{dx}\right)-u\left(\frac{dv}{dx}\right)}{v^2}$.
3. Chain Rule:- y=f(u), u=g(x)⇒ $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$ i.e., $y = f(g(x)) \Rightarrow \frac{dy}{dx} = f'(g(x)).g'(x)$
4. DERIVATIVES OF SOME STANDARD AND USEFUL FUNCTIONS:-

$\frac{d}{dx}(x^n) = nx^{n-1}$; $\frac{d}{dx}(k) = 0$; $\frac{d}{dx}(kx) = k$; $\frac{d}{dx}(kx^2) = 2kx$; $\frac{d}{dx}(kx^3) = 3kx^2$;

$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$; $\frac{d}{dx}.\frac{1}{x^n} = \frac{-n}{x^{n+1}}$; $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$; $\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = \frac{-1}{2x\sqrt{x}}$

$\frac{d}{dx}(ax+b) = a$; $\frac{d}{dx}(ax+b)^2 = 2(ax+b)a$; $\frac{d}{dx}\sqrt{ax+b} = \frac{2}{2\sqrt{ax+b}}$

$\frac{d}{dx}(logx) = \frac{1}{x}$; $\frac{d}{dx}log_a x = \frac{1}{x}log_a e$; $\frac{d}{dx}logf(x) = \frac{1}{f(x)}.f'(x)$

$\frac{d}{dx}e^x = e^x$; $\frac{d}{dx}e^{f(x)} = e^{f(x)}f'(x)$; $\frac{d}{dx}a^x = a^x log_e a$; $\frac{d}{dx}x^x = x^x(1+logx)$

TRIGONOMETRIC

$\frac{d}{dx}sinx = cosx$

$\frac{d}{dx}cosx = -sinx$

$\frac{d}{dx}tanx = sec^2x$

$\frac{d}{dx}cotx = -cosec^2x$

$\frac{d}{dx}secx = secx.tanx$

$\frac{d}{dx}cosecx = -cosecxcotx$

HYPERBOLIC

$\frac{d}{dx}sinhx = coshx$

$\frac{d}{dx}coshx = sinhx$

$\frac{d}{dx}tanhx = sech^2x$

$\frac{d}{dx}cothx = -cosec^2x$

$\frac{d}{dx}sechx = -sechxtanhx$

$\frac{d}{dx}cosechx = -cosechxtanhx$

INVERSE TRIGONOMETRIC

$\frac{d}{dx}sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$

$\frac{d}{dx}tan^{-1}x = \frac{1}{1+x^2}$

INVERSE HYPERBOLIC

$\frac{d}{dx}sinh^{-1}x = \frac{1}{\sqrt{1+x^2}}$

$\frac{d}{dx}cosh^{-1}x = \frac{1}{\sqrt{x^2-1}}$

$\frac{d}{dx}tanh^{-1}x = \frac{1}{1-x^2}$

$\frac{d}{dx}cot^{-1}x = \frac{-1}{1+x^2}$ $\frac{d}{dx}coth^{-1}x = \frac{1}{1-x^2}$

$\frac{d}{dx}sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}$ $\frac{d}{dx}sech^{-1}x = \frac{-1}{|x|\sqrt{1-x^2}}$

$\frac{d}{dx}cosec^{-1}x = \frac{-1}{|x|\sqrt{x^2-1}}$ $\frac{d}{dx}cosech^{-1}x = \frac{1}{|x|\sqrt{1+x^2}}$

Logarithm Differentiation" if $y = f(x)^{g(x)}$ then

$$logy = g(x)logf(x) \Rightarrow \frac{1}{y}\frac{dy}{dx} = g(x)\frac{1}{f(x)}f'(x) + logf(x)g'(x)$$

Derivative of a function w.r. to another function:

If $y = f(x), z = g(x)$ then $\frac{dy}{dz} = \frac{df}{dg} = \frac{f'(x)}{g'(x)}$

Parametric Differentiation:

If $x = f(x),\ y = g(t)$ then $\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$ and $\frac{d^2y}{d^2x} = \left[\frac{d}{dt}\left[\frac{dy}{dx}\right]\right]\left[\frac{dt}{dx}\right]$

Double differentiation:

$$f''(x) = \lim_{h\to 0}\frac{f'(x+n)-f'(x)}{h}$$

DIFFERENTIATION

2 MARKS LEVEL – I

1. Find the derivative of the function $f(x) = (x^2 - 3)(4x^3 + 1)$.

Sol: Formula: $\frac{d}{dx}(uv) = u.\frac{d}{dx}(v) + v.\frac{d}{dx}(u)$

Given $f(x) = (x^2 - 3)(4x^3 + 1)$

$f'(x) = \frac{d}{dx}(x^2 - 3)(4x^3 + 1)$

$= (x^2 - 3)\frac{d}{dx}(4x^3 + 1) + (4x^3 + 1)\frac{d}{dx}(x^2 - 3)$

$= (x^2 - 3)(12x^2) + (4x^3 + 1)(2x)$

$= 12x^4 - 36x^2 + 8x^4 + 2x$

$= 20x^4 - 36x^2 + 2x$

2. Find the derivative of $f(x) = e^x(x^2 + 1)$.

Sol: Given $f(x) = e^x(x^2 + 1)$

Applying uv formula, we have

$\Rightarrow f'(x) = e^x\frac{d}{dx}(x^2 + 1) + (x^2 + 1)\frac{d}{dx}(e^x)$

$= e^x(2x) + (x^2 + 1)e^x$

$= e^x(x^2 + 2x + 1) = e^x(x + 1)^2$

3. Find the Derivative of $5^x + logx + x^3e^x$.

Sol: $\frac{d}{dx}(5^x + logx + x^3e^x)$

$= 5^x log5 + \frac{1}{x} + x^3(e^x) + e^x(3x^2)$

4. If $y = e^{2x}.log(3x + 4)$ then find $\frac{dy}{dx}$.

Sol: Formula $\frac{d}{dx}(uv) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)$

Given $y = e^{2x}.\log(3x + 4)$

$\frac{dy}{dx} = e^{2x}\frac{d}{dx}[\log(3x + 4)] + \log(3x + 4)\frac{d}{dx}(e^{2x})$

$= e^{2x}\left[\frac{1}{3x+4}(3)\right] + \log(3x+4)\, e^{2x}.2$

$= e^{2x}\left[\frac{3}{3x+4} + 2\log(3x+4)\right]$

5. If $f(x) = e^{2x}logx,\ (x > 0)$ then find $f'(x)$.

Sol: Given $f(x) = e^{2x}logx$.

Applying uv formula and chain rule, we have

$\therefore f'(x) = e^{2x}\frac{d}{dx}[logx] + logx\frac{d}{dx}(e^{2x})$

$= e^{2x}\frac{1}{x} + logx(e^{2x})(2)$

$= e^{2x}\left(\frac{1}{x} + 2logx\right)$

6. Find the derivative of $e^x + sinx\, cosx$.

Sol: $\frac{d}{dx}(e^x + sinx\, cosx) = \frac{d}{dx}(e^x) + \frac{d}{dx}(sinx\, cosx)$

$= e^x + sinx\frac{d}{dx}(cosx) + cosx\frac{d}{dx}(sinx)$

$= e^x + sinx(-sinx) + cosx(cosx)$

$= e^x + (-sin^2x + cos^2x)$

$= e^x + (cos^2x - sin^2x)$

$= e^x + cos2x$ $[\therefore\ cos2\theta = cos^2\theta - sin^2\theta]$

7. If $f(x) = xe^x sinx$ then find $f'(x)$.

Sol: $\frac{d}{dx}(uvw) = uv\frac{d}{dx}(w) + uw\frac{d}{dx}(v) + vw\frac{d}{dx}(u)$

$\therefore\ f'(x) = -(xe^x sinx)$

$= xe^x\frac{d}{dx}(sinx) + xsinx\frac{d}{dx}(e^x) + e^x sinx\frac{d}{dx}(x)$

$= xe^x(cosx) + xsinx(e^x) + e^x sinx(1)$

$= xe^x cosx + xsinx(e^x) + e^x sinx$

$= e^x(xcosx + xsinx + sinx)$.

8. If $y = x^2e^x sinx$, then find $\frac{dy}{dx}$.

Sol: $\frac{d}{dx}(uvw) = uv\frac{d}{dx}(w) + uw\frac{d}{dx}(v) + vw\frac{d}{dx}(u)$

$\frac{d}{dx}(x^2e^x sinx) = x^2e^x\frac{d}{dx}(sinx) + x^2sinx\frac{d}{dx}(e^x) + e^x sinx\frac{d}{dx}(x^2)$

$= x^2e^x(cosx) + x^2sinx(e^x) + e^x sinx(2x)$

$= xe^x(xcosx + xsinx + 2sinx)$.

9. If $y = \frac{2x+3}{4x+5}$ then find $\frac{dy}{dx}$.

Sol: Formula $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v.\frac{d}{dx}(u) - u.\frac{d}{dx}(v)}{v^2}$

$\frac{dy}{dx} = \frac{(4x+5)\frac{d}{dx}(2x+3) - (2x+3)\frac{d}{dx}(4x+5)}{(4x+5)^2}$

$= \frac{(4x+5)(2) - (2x+3)(4)}{(4x+5)^2}$

$= \frac{8x+10-8x-12}{(4x+5)^2} = \frac{-2}{(4x+5)^2}$.

10. If $y = \frac{a-x}{a+x}, (x \neq -a)$ then find $\frac{dy}{dx}$.

Sol: Formula $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v.\frac{d}{dx}(u) - u.\frac{d}{dx}(v)}{v^2}$

$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{a-x}{a+x}\right) = \frac{(a+x)(-1) - (a-x)(1)}{(a+x)^2}$

$= \frac{-a-x-a+x}{(a+x)^2} = \frac{-2a}{(a+x)^2}$

11. If $f(x) = \sin(logx), (x > 0)$ then find $f'(x)$.

Sol: $\frac{d}{dx}\left(sin(logx)\right) = \cos(logx)\frac{d}{dx}(\log x)$

$= \cos(\log x)\frac{1}{x}$

$= \frac{\cos(\log x)}{x}$

12. Find the derivative of $\log(\sec x + \tan x)$.

Sol: We take $y = log(\sec x + \tan x)$, then

$\frac{dy}{dx} = \frac{d}{dx}\log(secx + tanx)$

$= \frac{1}{secx+tanx}\ \frac{d}{dx}(secx + tanx)$

$= \frac{1}{\sec x+\tan x}(\sec x + \tan x +\ sec^2 x)$

$= \frac{secx(tanx+secx)}{secx+tanx}\ = secx$

13. Find the derivative of $y = e^{sin^{-1}x}$.

Sol: $\frac{dy}{dx} = e^{sin^{-1}x}\frac{d}{dx}(sin^{-1}(x))$

$= e^{sin^{-1}x}\left[\frac{1}{\sqrt{1-x^2}}\right] = \frac{e^{sin^{-1}x}}{\sqrt{1-x^2}}$

14. If $y = sin^{-1}(cosx)$ then find $\frac{dy}{dx}$.

Sol: Given $y = sin^{-1}(cosx)$, then

$\frac{dy}{dx} = \frac{d}{dx}sin^{-1}(cosx) = \frac{1}{\sqrt{1-(\cos x)^2}}\frac{d}{dx}(cosx)$

$= \frac{-sinx}{\sqrt{1-cos^2x}} = \frac{-sinx}{sinx} = -1$

15. Find the derivative of $\log(\sin(\log(x))$.

Sol: We take $y = \log(\sin(\log(x))$, then

$\frac{dy}{dx} = \frac{d}{dx}log(\sin(logx))$

$= \frac{1}{\sin(logx)}\ \frac{d}{dx}\sin(logx)$

$= \frac{1}{\sin(logx)}\cos(logx).\frac{d}{dx}(logx)$

$= \frac{\cos(logx)}{\sin(logx)}.\frac{1}{x}\ = \frac{\cot(logx)}{x}$

16. If $y = e^{asin^{-1}x}$ then find $\frac{dy}{dx}$.

Sol: $\frac{dy}{dx} = e^{asin^{-1}x}\frac{d}{dx}(asin^{-1}(x))$

$= e^{asin^{-1}x}.a\frac{1}{\sqrt{1-x^2}} + sin^{-1}x(0)$

$= e^{asin^{-1}x} . \frac{a}{\sqrt{1-x^2}}$

$= \frac{e^{asin^{-1}x} a}{\sqrt{1-x^2}} = \frac{ay}{\sqrt{1-x^2}}$

17. If $y = tan^{-1}(logx)$ then find $\frac{dy}{dx}$.

Sol: Given $y = tan^{-1}(logx)$

$\frac{dy}{dx} = \frac{d}{dx}\left(tan^{-1}(logx)\right)$

$= \frac{1}{1+(logx)^2} \frac{d}{dx}(logx)$

$= \left[\frac{1}{1+(logx)^2}\right]\frac{1}{x}$

18. Find the derivative of $tan^{-1}\left[\frac{2x}{1-x^2}\right]$.

Sol: We take $x = tan\theta$ then, $\theta = tan^{-1}x$

$\therefore \ tan^{-1}\left[\frac{2x}{1-x^2}\right] = tan^{-1}\left(\frac{2tan\theta}{1-tan^2\theta}\right)$

$= tan^{-1}tan2\theta = 2\theta = 2tan^{-1}x = \frac{2}{1+x^2} \quad \left[\because \quad \frac{d}{dx}\left(2tan^{-1}x = \frac{2}{1+x^2}\right)\right]$

19. Find the derivative of $sin^{-1}(3x - 4x^3)$.

Sol: We take $x = sin\theta$ then, $\theta = sin^{-1}x$

$\therefore \ sin^{-1}(3x - 4x^3) = sin^{-1}[3(sin\theta) - 4(sin\theta)^3]$

$= sin^{-1}(sin3\theta)$

$= 3\theta$

$= 3(sin^{-1}x)$

$\therefore \ \frac{d}{dx}(3sin^{-1}x) = 3\frac{d}{dx}sin^{-1}x = 3\left[\frac{1}{\sqrt{1-x^2}}\right] = \frac{3}{\sqrt{1-x^2}}$

20. Find the derivative of $sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Sol: We take $x = tan\theta$ then, $\theta = tan^{-1}x$

$\therefore \ sin^{-1}\left(\frac{2x}{1+x^2}\right) = sin^{-1}\left(\frac{2tan\theta}{1+tan^2\theta}\right)$

$= sin^{-1}(sin2\theta) = 2\theta = 2(tan^{-1}x) \qquad [\because \quad x = tan\theta \Rightarrow \theta = tan^{-1}x]$

$\therefore \frac{d}{dx}(2tan^{-1}x) = 2\frac{d}{dx}tan^{-1}x$

$= 2\left(\frac{1}{1+x^2}\right) = \frac{2}{1+x^2}$

21. Find the derivative of $cos^{-1}(4x^3 - 3x)$.

Sol: We take $x = cos\theta$ then, $\theta = cos^{-1}x$

$\therefore cos^{-1}(4x^3 - 3x) = cos^{-1}(4cos^3\theta - 3cos\theta)$

$= cos^{-1}(cos3\theta)$

$= 3\theta$

$= 3(cos^{-1}x)$

$\therefore \frac{d}{dx}(3cos^{-1}x) = 3\left[\frac{-1}{\sqrt{1-x^2}}\right] = \frac{-3}{\sqrt{1-x^2}}$

22. Find the derivative of $y = \sqrt{2x-3} + \sqrt{7-3x}$.

Sol: Given $y = \sqrt{2x-3} + \sqrt{7-3x}$, then

$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{2x-3}\right) + \sqrt{7-3x}$

$= \frac{2}{2\sqrt{2x-3}}\ (2) + \frac{1}{2\sqrt{7-3x}}(-3)$

$= \frac{1}{\sqrt{2x-3}} - \frac{3}{2\sqrt{7-3x}}$

23. Find the derivative of 7^{x^3+3x} .

Sol: $\frac{d}{dx}a^{f(x)} = a^{f(x)}loga. f'(x)$

$\therefore \frac{d}{dx}\left(7^{x^3+3x}\right) = 7^{x^3+3x}(\log 7).\frac{d}{dx}(x^3+3x) = 7^{x^3+3x}(\log 7)(3x^2+3)$

24. If $y = (cot^{-1}x^3)^2$ then find $\frac{dy}{dx}$.

Sol: $\frac{dy}{dx} = \frac{d}{dx}(cot^{-1}x^3)^2 = 2cot^{-1}x^3\frac{d}{dx}(cot^{-1}x^3)$ [$\because$ from chain rule]

$= 2cot^{-1}x^3\ \left(\frac{-1}{1+(x^3)^2}\right)3x^2 = \frac{-6x^2cot^{-1}x^3}{1+x^6}$

25. Find $\frac{dy}{dx}$ if $2x^2 - 3xy + y^2 + x + 2y - 8 = 0$.

Sol: Given that $2x^2 - 3xy + y^2 + x + 2y - 8 = 0$; differentiating with respect to x, we have

$4x - 3\left[x\frac{dy}{dx} + y(1)\right] + 2y\frac{dy}{dx} + 1 + 2\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx}(2y - 3x + 2) = 3y - 4x - 1 \;\Rightarrow \frac{dy}{dx} = \frac{3y-4x-1}{2y-3x+2}$

26. Find $\frac{dy}{dx}$ if $x = acos^3t,\ y = asin^3t.$

Sol: Given $x = acos^3t$ o differentiating w. r. to 't' we get,

$\frac{dx}{dt} = \frac{d}{dt}\ (acos^3t) = a.3cos^2t\ \frac{d}{dt}(cost)$

$= a.3cos^2t\ (-sint)$

$= -3a\ cos^2t.(sint)$

$\frac{dy}{dt} = \frac{d}{dt}\ (asin^3t) = a.3sin^2t\ \frac{d}{dt}(sint)$

$= a.3sin^2t\ (cost)$

$= 3a\ sin^2t.cost$

$\therefore\ \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3a\ sin^2t\ (cost)}{-3a\ cos^2t\ (sint)} = -\frac{sint}{cost} = -tant$

27. Find the derivative of x^x.

Sol: We take $y = x^x \Rightarrow logy = logx^x$

$\Rightarrow logy = x\ logx$, on differentiation w. r. to $'x'$ we get

$\frac{1}{y}\ \frac{dy}{dx} = x\left(\frac{1}{x}\right) + logx(1) \;\Rightarrow\; \frac{dy}{dx} = y(1 + logx) \Rightarrow \frac{dy}{dx} = x^x(1 + logx)$

28. If $y = x^4 + tanx$ then find y'' .

Sol: Given $y = x^4 + tanx$

= differentiation with respect to x we get

$y' = 4x^3 + secx$

Again differentiation with respect to x we get

$y'' = 12x^2 + 2secx(secx\ tanx)$

$= 12x^2 + 2sec^2x\ tanx$

29. If $y = ae^{nx} + be^{-nx}$, then prove that $y'' = n^2 y$.

Sol: Given $y = ae^{nx} + be^{-nx}$; on differentiation with respect to x we get

$y' = ae^{nx}(n) + be^{-nx}(-n)$ on differentiation again with respect to x, we get

$y'' = ae^{nx}(n)(n) + be^{-nx}(-n)(-n)$

$y'' = n^2 ae^{nx} + n^2 be^{-nx} = n^2(ae^{nx} + be^{-nx}) = n^2 y$

$\therefore\ y'' = n^2 y$

30. If $f(x) = 2x^2 + 3x - 5$ then prove that $f'(0) + 3f'(-1) = 0$

Sol: Given $f(x) = 2x^2 + 3x - 5 \Rightarrow f'(x) = 4x + 3$

Then $f'(0) = 0 + 3$ and $f'(-1) = -4 + 3 = -1$

$\therefore\ f'(0) + 3f^1(-1) = 3 + 3(-1) = 3 - 3 = 0$

31. If $f(x) = 1 + x + x^2 + \cdots + x^{100}$, then find $f'(1)$.

Sol: $f(x) = 1 + x + x^2 + \cdots + x^{100} \Rightarrow f'(x) = 1 + 2x + 3x^2 + \cdots + 100x^{99}$

$\Rightarrow f'(1) = 1 + 2 + 3 + \cdots + 100 = \frac{100\times101}{2} = 5050\quad \left[\because\ \sum n = \frac{n(n+1)}{2}\right]$

LEVEL – 1, 4 Marks

1. Find the derivative of $\sin 2x$ from the first principle.

Sol: We take $f(x) = sin2x$, then

$f(x+h) = sin2(x+h) = \sin(2x+2h)$

From the first principle,

$$f'(x) = \lim_{h\to0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to0}\frac{\sin(2x+2h)-\sin 2x}{h}$$

$$= \lim_{h\to0}\frac{1}{h}\left[2\cos\left(\frac{(2x+2h)+2x}{2}\right)sin\left(\frac{(2x+2h)-2x}{2}\right)\right]\ \left(\because sinC - sinD = 2cos\left(\frac{C+D}{2}\right)sin\left(\frac{C-D}{2}\right)\right)$$

$$= 2\lim_{h\to0}\frac{1}{h}\left[cos\left(\frac{4x+2h}{2}\right)sin\left(\frac{2h}{2}\right)\right]$$

$$= 2\lim_{h\to0}\frac{1}{h}cos\left(\frac{(2(2x+h))}{2}\right)\sin(h)$$

$= 2\lim_{h\to0}\frac{1}{h}\cos(2x+h)\sin(h)$

$= 2\lim_{h\to0}\cos(2x+h)\,.\lim_{h\to0}\frac{sinh}{h}$

$= 2\cos(2x+0)\,(1) = 2cos2x$

2. Find the derivative of $\cos ax$ from the first principle.

Sol: We take $f(x) = cosax$, then

$f(x+h) = cosa(x+h) = cos(ax+ah)$

From the first principle,

$f'(x) = \lim_{h\to0}\frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to0}\frac{\cos(ax+ah)-\sin ax}{h}$

$= \lim_{h\to0}\frac{1}{h}\left[-2\sin\left(\frac{(ax+ah)+ax}{2}\right)sin\left(\frac{(ax+ah)-ax}{2}\right)\right]\ \left(\because cosC - cosD = -2sin\left(\frac{C+D}{2}\right)sin\left(\frac{C-D}{2}\right)\right)$

$= -2\lim_{h\to0}\frac{1}{h}\left[sin\left(\frac{2ax+ah}{2}\right)sin\left(\frac{ah}{2}\right)\right]$

$= -2\lim_{h\to0}\frac{1}{h}\left(sin\left(ax+\frac{ah}{2}\right)\sin(\frac{ah}{2})\right)$

$= -2\lim_{h\to0}\frac{1}{h}\sin\left[ax+\frac{ah}{2}\right]\lim_{h\to0}\frac{sin\left(\frac{ah}{2}\right)}{h}$

$= -2\sin(ax+0)\left(\frac{a}{2}\right) = -2\,sinax\left(\frac{a}{2}\right) = -a\sin ax$

3. Find the derivative of $\tan 2x$ from the first principle.

Sol: We take $f(x) = tan2x$, then

$f(x+h) = tan2(x+h) = tan(2x+2h)$

From the first principle,

$f'(x) = \lim_{h\to0}\frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to0}\frac{\tan(2x+2h)-tan\,2x}{h}$

$= \lim_{h\to0}\frac{1}{h}\left[\frac{\sin(2x+2h)}{\cos(2x+2h)} - \frac{\sin(2x)}{\cos(2x)}\right]$

$= \lim_{h\to 0} \frac{1}{h}\left[\frac{\sin(2x+2h)\cos(2x)-\cos(2x+2h)\sin(2x)}{\cos(2x+2h)\cos(2x)}\right]$

$= \lim_{h\to 0} \frac{1}{h} \frac{sin[(2x+2h)-2x]}{\cos(2x+2h)\cos(2x)}$

$[\because sinAcosB - cosAsinB = \sin(A-B)]$

$= \lim_{h\to 0} \frac{sin2h}{h} . \lim_{h\to 0} \frac{1}{\cos(2x+2h)\cos(2x)}$

$= 2.\frac{1}{cos^2(2x)} = 2sec^2(2x)$

4. Find the derivative of $\cot x$ from the first principle.

Sol: We take $f(x) = cotx$, then

$f(x+h) = cot(x+h)$

From the first principle,

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0} \frac{\cot(x+h)-cotx}{h}$

$= \lim_{h\to 0} \frac{1}{h}\left[\frac{\cos(x+h)}{\sin(x+h)} - \frac{\cos(x)}{\sin(x)}\right]$

$= \lim_{h\to 0} \frac{1}{h}\left[\frac{\cos(x+h).sinx-\sin(x+h).cosx}{\sin(x+h).sinx}\right]$

$= \lim_{h\to 0} \frac{1}{h}\left[\frac{-\sin((x+h)-x)}{\sin(x+h).sinx}\right]$

$[\because cosA\ sinB - sinA\ cosB = -\sin(A-B)]$

$= \lim_{h\to 0} \frac{1}{h}\left[\frac{-sinh}{\sin(x+h)sinx}\right]$

$= \lim_{h\to 0}\left[\frac{-\sin h}{h}\right] . \lim_{h\to 0} \frac{1}{\sin(x+h)sinx}$

$= -1\left[\frac{1}{sinx.sinx}\right] = -cosec^2 x$

5. Find the derivative of $cos^2 x$ from the first principle.

Sol: We take $f(x) = cos^2 x$, then

$f(x+h) = cos^2(x+h)$

From the first principle,

$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0}\frac{cos^2(x+h)-cos^2x}{h}$

$= \lim_{h\to 0}\frac{-sin[(x+h)+x]sin[(x+h)-x]}{h}$ $[\because\ cos^2A - cos^2B = -\sin(A+B)\sin(A-B)]$

$= \lim_{h\to 0}\frac{-\sin(2x+h)\sin(h)}{h} = -\lim_{h\to 0}\sin(2x+h).\lim_{h\to 0}\frac{\sin(h)}{h}$

$= -\sin(2x+0)\,(1) = sin2x$

6. Find the derivative of $sec3x$ using first principle.

Sol: We take $f(x) = sec3x$, then

$f(x+h) = sec3(x+h) = \sec(3x+3h)$

From the first principle,

$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0}\frac{\sec(3x+3h)-sec3x}{h}$

$= \lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(3x+3h)} - \frac{1}{\cos 3x}\right]$

$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\cos 3x-\cos(3x+3h)}{\cos(3x+3h)cos3x}\right]$

$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\frac{2\sin(3x+(3x+3h))}{2}sin\frac{(3x+3h)-3x}{2}}{\cos(3x+3h)cos3x}\right]$

$\left[\because cosC - cosD = +2sin\left(\frac{C+D}{2}\right)sin\left(\frac{D-C}{2}\right)\right]$

$= \lim_{h\to 0}\frac{1}{h}\left[\frac{2sin\left(\frac{6x}{2}+\frac{3h}{2}\right)sin\left(\frac{3h}{2}\right)}{\cos(3x+3h)cos3x}\right]$

$= 2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin\left(3x+\frac{3h}{2}\right)sin\left(\frac{3h}{2}\right)}{\cos(3x+3h)cos3x}\right] = 2\lim_{h\to 0}\frac{1}{h}\frac{sin\left(3x+\frac{3h}{2}\right)}{\cos(3x+3h)cos3x}.\lim_{h\to 0}\frac{sin\left(\frac{3h}{2}\right)}{h}$

$= 2.\frac{\sin(3x+0)}{\cos(3x+0)\cos(3x)}.\left(\frac{3}{2}\right) = 3\,\frac{sin3x}{cos3x.cos3x} = 3\,\frac{1}{cos3x}\left[\frac{sin3x}{cos3x}\right]$

$= secx.tan3x$

7. Find the derivative of $x\ sinx$ from the first principle.

Sol: We take $f(x) = x\ sinx$, then

$f(x+h) = (x+h)\ sin(x+h) = [x(\sin(x+h)) + \text{h}\ (\sin(x+h))]$

From the first principle,

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0} \frac{x[\sin(x+h)]+h9\sin(x+h)]-x\ sinx}{h}$

$= \lim_{h\to 0} \frac{x[\sin(x+h)-sinx]+h\sin(x+h)}{h}$

$= \lim_{h\to 0} \frac{x}{h}[\sin(x+h) - sinx] + \lim_{h\to 0} \frac{h\ sin(x+h)}{h}$

$= \lim_{h\to 0} \frac{x}{h}\left[2cos\frac{(x+h)+x}{2}.sin\frac{(x+h)-x}{2}\right] + \lim_{h\to 0} \sin(x+h)$

$= \lim_{h\to 0} \frac{x}{h}\left[2cos\left[\frac{2x}{2}+\frac{h}{2}\right] sin\frac{h}{2}\right] + \lim_{h\to 0} \sin(x+h)$

$= \lim_{h\to 0} \frac{x}{h}\left[2\cos\left(x+\frac{h}{2}\right) sin\left(\frac{h}{2}\right)\right] + \lim_{h\to 0} \sin(x+h)$

$= \lim_{h\to 0} 2xcos\left(x+\frac{h}{2}\right) \lim_{h\to 0}\left[\frac{sin\frac{h}{2}}{h}\right] + \lim_{h\to 0} \sin(x+h) = 2xcos(x+0).\left[\frac{1}{x}\right] + sinx$

$= x\ cosx + sinx$

8. Find the derivative of x^3 from the first principle.

Sol: We take $f(x) = x^3$, then

$f(x+h) = (x+h)^3$

From the first principle,

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0} \frac{(x^3+h^3+3x^2h+3xh^2)-x^3}{h}$ $[\because\ (a+b)^3 = a^3+b^3+3a^2b+3ab^2]$

$= \lim_{h\to 0} \frac{h^3+3x^2h+3xh^2}{h} = \lim_{h\to 0}\left(\frac{h(h^2+3x^2+3xh)}{h}\right)$

$= \lim_{h\to 0}(3x^2+3xh+h^2) = 3x^2+3x(0)+(0)^2 = 3x^2$

Hence proved that derivative of x^3 is $3x^2$.

9. Find the derivative of $\sqrt{x+1}$ from the first principle.

Sol: We have $f(x) = \sqrt{x+1}$, then $f(x+h) = \sqrt{x+h+1}$

From the first principle, $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0} \frac{1}{h}\left(\sqrt{x+h+1} - \sqrt{x+1}\right)$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{(\sqrt{x+h+1}-\sqrt{x+1})(\sqrt{x+h+1}+\sqrt{x+1})}{\sqrt{x+h+1}+\sqrt{x+1}}$$

$$= \lim_{h\to 0} \frac{(x+h+1)-(x+1)}{\sqrt{x+h+1}+\sqrt{x+1}} \quad \left[\because \ (\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = a-b\right]$$

$$= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h+1}+\sqrt{x+1})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h+1}+\sqrt{x+1}}$$

$$= \frac{1}{\sqrt{x+0+1}+\sqrt{x+1}} = \frac{1}{2\sqrt{x+1}}$$

Hence proved that derivative of $\sqrt{x+1}$ is $\frac{1}{2\sqrt{x+1}}$.

10. If $x^y = e^{x-y}$, then show that $\frac{dy}{dx} = \frac{logx}{(1+logx)^2}$

Sol: Given $x^y = e^{x-y}$,

Applying log on both sides, $logx^y = loge^{x-y} = (x-y)loge \ \ [\because logx^n = nlogx]$

$\Rightarrow y\, logx = (x-y).1 \quad [\because \ \ loge = 1]$

$\Rightarrow y\, logx + y = x \ \Rightarrow y(logx+1) = x \Rightarrow y = \frac{x}{lox+1}$

Differentiating w.r.t. x on both sides

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{x}{logx+1}\right] \Rightarrow \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{(\text{logx}+1)\frac{d}{dx}(x) - \frac{d}{dx}(logx+1)x}{(logx+1)^2} \quad \left[\because \left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{(logx+1)-x\left[\frac{1}{x}\right]}{(logx+1)^2} = \frac{logx+1-1}{(1+logx)^2}$$

$$\therefore \ \frac{dy}{dx} = \frac{logx}{(1+logx)^2}$$

11. If $siny = xsin(a+y)$, then show that $\frac{dy}{dx} = \frac{sin^2(a+y)}{sina}$ (a is not a multiple of π)

Sol: Given $siny = xsin(a+y)$

$\Rightarrow x = \frac{siny}{\sin(a+y)}$

Diff. w.r.to y

$\frac{dx}{dy} = \frac{\sin(a+y)(\sin y)\prime - siny\,(\sin(a+y))\prime}{(sin^2(a+y))}$ $\left[\because \left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2}\right]$

$= \frac{\sin(a+y)cosy - sinycos(a+y)}{sin^2(a+y)}$ $\left[\because \frac{d}{dx}(sinx) = cosx\right]\left[\because \frac{d}{dx}(cosx) = -sinx\right]$

$\Rightarrow \frac{dx}{dy} = \frac{\sin(a+y-y)}{sin^2(a+y)}$ $[\because sinAcosB - cosAsinB = \sin(A-B)]$

$\Rightarrow \frac{dx}{dy} = \frac{sina}{sin^2(a+y)}$

$\Rightarrow \therefore \frac{dy}{dx} = \frac{sin^2(a+y)}{sina}.$

12. If $f(x) = tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$, $g(x) = tan^{-1}x$ then, differentiate $f(x)$ with respect to $g(x)$.

Sol: Given $f(x) = tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$

Put $x = tan\theta$ then $\theta = tan^{-1}x$

$f(x) = tan^{-1}\left[\frac{\sqrt{1+(tan^2\theta)}-1}{tan\theta}\right] = tan^{-1}\left[\frac{\sqrt{sec^2\theta}-1}{tan\theta}\right]$ $[\because 1 + tan^2\theta = sec^2\theta]$

$= tan^{-1}\left[\frac{sec\theta-1}{tan\theta}\right] = tan^{-1}\left[\frac{\frac{1}{cos\theta}-\frac{1}{1}}{\frac{sin\theta}{cos\theta}}\right]$ $\left[\because sec\theta = \frac{1}{cos\theta}\right]\left[\because tan\theta = \frac{sin\theta}{cos\theta}\right]$

$= tan^{-1}\left[\frac{\frac{1-cos\theta}{cos\theta}}{\frac{sin\theta}{cos\theta}}\right]$

$= tan^{-1}\left[\frac{1-cos\theta}{sin\theta}\right] = tan^{-1}\left[\frac{2sin^2\frac{\theta}{2}}{2sin\frac{\theta}{2}cos\frac{\theta}{2}}\right]$ $\left[\because 1 - cosA = 2sin^2\frac{A}{2}\right]$

$= tan^{-1}\left[\frac{sin\frac{\theta}{2}}{cos\frac{\theta}{2}}\right] = tan^{-1}\left(tan\frac{\theta}{2}\right)$ $\left[\because sinA = 2sin\frac{A}{2}cos\frac{A}{2}\right]$

$=\frac{\theta}{2} = \frac{tan^{-1}(x)}{2}$ $[\because tan^{-1}(tan\theta) = \theta]$

$\Rightarrow f'(x) = \frac{1}{2} \times \frac{1}{1+x^2}$ $\left[\because \frac{d}{dx}(tan^{-1}x) = \frac{1}{1+x^2}\right]$

$\Rightarrow f'(x) = \frac{1}{2(1+x^2)}$

$g(x) = tan^{-1}(x) \Rightarrow g'(x) = \frac{1}{1+x^2}$

$\frac{f\prime(x)}{g\prime(x)} = \frac{\frac{1}{1(1+x^2)}}{\frac{1}{1+x^2}} = \frac{1}{2}$

13. If $y = x^4$ then show that $\frac{dy}{dx} = \frac{y^2}{x(1-ylogx)} = \frac{y^2}{x(1-ylogx)}$

Sol: Given $y = x^4$

Applying log on both sides, $logy = logx^4 = y\, logx$ …… (1) $[\because \ logx^n = nlogx]$

Differentiating w.r. to x on both sides

$\frac{d}{dx}(logy) = \frac{d}{dx}(y\, logx)$

$\frac{1}{y} \times \frac{dy}{dx} = y\frac{d}{dx}(logx) + logx\frac{d}{dx}(y)$ $\left[\because \frac{d}{dx}\big(logf(y)\big) = \frac{1}{f(y)}f'(y)\right]\left[\because \left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2}\right]$

$\Rightarrow \frac{1}{y} \times \frac{dy}{dx} = y \times \frac{1}{x} + logx\frac{dy}{dx} = \frac{dy}{dx}\left[\frac{1}{y} - \frac{logx}{1}\right] = \frac{y}{x}$ $\quad [\because \ \frac{d}{dx}(logx) = \frac{1}{x}]$

$\Rightarrow \frac{dy}{dx}\left[1 - ylog\frac{x}{y}\right] = \frac{y}{x} \Rightarrow \frac{dy}{dx} = \frac{y}{x} \times \frac{y}{1-ylogx} = \frac{y^2}{x(1-y\, logx)}$

$\Rightarrow \frac{dy}{dx} = \frac{y^2}{x(1-logy)}$

14. Find $\frac{dy}{dx}$ for the functions, $x = a(cost + t\, sint), y = a(sint - t\, cost)$.

Sol: Given $x = a(cost + \ t\, sint)$

Differentiating with respect to x on the both sides

$\Rightarrow \frac{dx}{dt} = a(cost)' + t(sint)' + sint\ (t)'$ $\quad [\because (uv)' = uv' + vu']$

$\frac{dx}{dt} = a(-sint + sint.(1) + t(cost))$ $\quad \left[\because \frac{d}{dx}(cosx) = -sinx\right]\left[\because \frac{d}{dx}(sinx) = cosx\right]$
$\left[\frac{d}{dx}(x) = 1\right]$

$\frac{dx}{dt} = a(-sint + sint + t\, cost) \Rightarrow \frac{dx}{dt} = atcost$

$y = a(sint - t\, cost)$

Differentiating with respect to "t" on both sides

$\Rightarrow \frac{dy}{dt} = a(cost - cost(1) + t(-sint))$ $\quad [\because (uv)' = uv' + vu']$

$\frac{dy}{dt} = a(cost - cost + t\ sint)$ $\quad \left[\because\ \frac{d}{dx}(cosx) = -sinx\right]\left[\because\ \frac{d}{dx}(sinx) = cosx\right]\left[\frac{d}{dx}(x) = 1\right]$

$\Rightarrow \frac{dy}{dt} = atsint$

$\therefore\ \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{atsint}{atcost} = tant$ $\quad \because \frac{sin\theta}{cos\theta} = tan\theta$

$\Rightarrow \frac{dy}{dx} = tant$

15. If $x^{2/3} + y^{2/3} = a^{2/3}$ then $\frac{dy}{dx} = -\sqrt[3]{\left[\frac{y}{x}\right]}$.

Sol: Given $x^{2/3} + y^{2/3} = a^{2/3}$

Differentiating with respect to $'x'$

$\frac{2}{3}x^{2/3-1} + \frac{2}{3}y^{2/3-1}.\frac{dy}{dx} = 0$ $\quad \left[\because\ \frac{d}{dx}(x^n) = nx^{n-1}\right]\left[\because\ \frac{d}{dx}(y)^n = ny^{n-1}.\frac{dy}{dx}\right]\left[\because\ \frac{d}{dx}(x) = 0\right]$

$\Rightarrow \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}.\frac{dy}{dx} = 0 \Rightarrow \frac{2}{3}y^{-1/3}.\frac{dy}{dx} = -\frac{2}{3}x^{-1/3}$

$\Rightarrow \frac{dy}{dx} = \frac{-\frac{2}{3}x^{-1/3}}{\frac{2}{3}y^{-1/3}} = \frac{-y^{1/3}}{x^{1/3}} = -\left[\frac{y}{x}\right]^{1/3} = -\sqrt[3]{\left[\frac{y}{x}\right]}$ $\quad \left[\because\ \frac{a^m}{b^m} = \left[\frac{a}{b}\right]^m\right]\ \left[\because\ \frac{1}{x^m} = \sqrt[m]{x}\right]$

16. If $y = acosx + (b + 2x)sinx$, then show that $y" + y = 4\ cosx$.

Sol: Given $y = acosx + (b + 2x)sinx$

Differentiation with respect to x on both sides

$y' = a(cosx)' + (b + 2x)(sinx)' + sinx(b + 2x)'$ $\quad [\because (uv)' = uv' + vu']$

$\Rightarrow y' = a(-sinx) + (b + 2x)cosx + sinx(2)$ $\quad \left[\frac{d}{dx}(cosx) = -sinx\right]\left[\frac{d}{dx}(sinx) = cosx\right]$

$\Rightarrow y' = -asinx + b + 2xcosx + 2sinx$

Again differentiation with respect to x on both sides

$\Rightarrow y" = -acosx + (b + 2x)(-sinx) + cosx(2) + 2cosx$ $\quad [\because (uv)' = uv' + vu']$

$\Rightarrow y" = -acosx - (b + 2x)sinx + 4cosx$

$\Rightarrow y" = -[acos + (b + 2x)sinx] + 4cosx$

$\Rightarrow y" = -y + 4cosx \Rightarrow y" + y = 4\ cosx$

17. If $x = 3cost - 2cos^3t$, $y = 3sint - 2sin^3t$, then find $\frac{dy}{dx}$.

Sol: Given $x = 3cost - 2cos^3t$, $y = 3sint - 2sin^3t$

Now, $x = 3cost - 2cos^3t$

Differentiation with respect to 't' we get

$\frac{dx}{dt} = -3sint - 2.3cos^2t(-sint)$ $\left[\because \frac{d}{dx}(x^n) = nx^{n-1}\right]\left[\because \frac{d}{dx}(cosx) = -sinx\right]$

$\Rightarrow \frac{dx}{dt} = 6sintcos^2t - 3sint \Rightarrow \frac{dx}{dt} = 3sint(2cos^2t - 1)$

$\Rightarrow \frac{dx}{dt} = 3sint\ cos2t$

Also $y = 3sint - 2sin^3t$

Differentiation with respect to 't' we get

$\frac{dy}{dt} = 3cost - 2.3sin^2t(cost)$ $\left[\because \frac{d}{dx}(x^n) = nx^{n-1}\right]\left[\because \frac{d}{dx}(sinx) = cosx\right]$

$\Rightarrow \frac{dy}{dt} = 3cost - 6costsin^2t \Rightarrow \frac{dy}{dt} = 3cost(t - 2sin^2t)$

$\Rightarrow \frac{dy}{dt} = 3cost\ cos2t$ $[\because 1 - 2sin^2A = cos2A]$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3costcos2t}{3sintcot2t} = \cot t$

LEVEL – I, 7Marks

1. If $\sqrt{1 - x^2} + \sqrt{1 - y^2} = a(x - y)$ then show that $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$.

Sol: Given$\sqrt{1 - x^2} + \sqrt{1 - y^2} = a(x - y)$

Put $x = sin\theta_1$ and $y = sin\theta_2 \Rightarrow \theta_1 = sin^{-1}x$, $\theta_2 = sin^{-1}y$

Now $\sqrt{1 - sin^2\theta_1} + \sqrt{1 - sin^2\theta_2} = a(sin\theta_1 - sin\theta_2)$

$\Rightarrow \sqrt{cos^2\theta_1} + \sqrt{cos^2\theta_2} = a(sin\theta_1 - sin\theta_2)$ $[\because 1 - sin^2\theta = cos^2\theta]$

$\Rightarrow cos\theta_1 + cos\theta_2 = a(sin\theta_1 - sin\theta_2)$ $[\because \sqrt{a^2} = |a|]$

$\Rightarrow 2cos\left[\frac{\theta_1+\theta_2}{2}\right]cos\left[\frac{\theta_1-\theta_2}{2}\right] = a2cos\left[\frac{\theta_1+\theta_2}{2}\right]sin\left[\frac{\theta_1-\theta_2}{2}\right]$

$\Rightarrow \frac{cos\left[\frac{\theta_1-\theta_2}{2}\right]}{sin\left[\frac{\theta_1-\theta_2}{2}\right]} = a \Rightarrow cot\left[\frac{\theta_1-\theta_2}{2}\right] = a$ $\left[\because \frac{cos\theta}{sin\theta} = cot\theta\right]$

$\Rightarrow \frac{\theta_1-\theta_2}{2} = cot^{-1}(a) \Rightarrow \theta_1 - \theta_2 = 2cot^{-1}(a)$ $\quad [\because \theta_1 = sin^{-1}x;\ \theta_2 = sin^{-1}y]$

$\Rightarrow sin^{-1}x - sin^{-1}y = 2cot^{-1}(a)$

Differentiating with respect to x

$\frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}} . \frac{dy}{dx} = 0$ $\quad \left[\because \frac{dy}{dx}(sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}\right] \left[\because \frac{d}{dx}(k) = 0\right]$

$\Rightarrow \frac{1}{\sqrt{1-y^2}} . \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \Rightarrow \frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} = \sqrt{\frac{1-y^2}{1-x^2}}$

2. If $y = tan^{-1}\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right]$for $0 < |x| < 1$, find $\frac{dy}{dx}$.

Sol: Given $y = tan^{-1}\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right]$

Substitute $x^2 = cos2\theta \Rightarrow \frac{1}{2}\ cos^{-1}x^2$

We get $y = tan^{-1}\left[\frac{\sqrt{1+cos2\theta}+\sqrt{1-cos2\theta}}{\sqrt{1+cos2\theta}-\sqrt{1-cos2\theta}}\right]$

$= tan^{-1}\left[\frac{\sqrt{2cos^2\theta}+\sqrt{2sin^2\theta}}{\sqrt{2cos^2\theta}-\sqrt{2sin^2\theta}}\right]$ $\quad [\because 1 + cos2A = 2cos^2A]\ [\because 1 - cos2A = 2sin^2A]$

$y = tan^{-1}\left[\frac{\sqrt{2}(cos\theta+sin\theta)}{\sqrt{2}(cos\theta-sin\theta)}\right] = tan^{-1}\left[\frac{cos\theta(1+tan\theta)}{cos\theta(1-tan\theta)}\right]$

$= tan^{-1}\left[tan\left(\frac{\pi}{4}+\theta\right)\right] = \frac{\pi}{4}+\theta$ $\quad \left[\because \frac{1+tan\theta}{1-tan\theta} = tan\left(\frac{\pi}{4}+\theta\right)\right] [tan^{-1}(tan\theta) = \theta]$

$\therefore y = \frac{\pi}{4} + \frac{1}{2}cos^{-1}(x^2)$

Differentiating with respect to $'x'$ we get

$\frac{dy}{dx} = 0 + \frac{1}{2} \times \frac{-1}{\sqrt{1-x^4}} \times 2x$

$= \frac{-x}{\sqrt{1-x^4}}$

3. If $y = x^{tanx} + (sinx)^{cosx}$, find $\frac{dy}{dx}$.

Sol: Given $y = x^{tanx} + (sinx)^{cosx}$

Let $y_1 = x^{tanx}$ and $y_2 = (sinx)^{cosx}$ then $y = y_1 + y_2$

Differentiating with respect to $'x'$ on both sides

$\frac{dy}{dx} = \frac{dy_1}{dx} + \frac{dy_2}{dx}$ ---------(1)

$y_1 = x^{tanx}$

Applying log on both sides, $logy_1 = log(x^{tanx})$

$\Rightarrow logy_1 = tanx.logx$ $[\because logx^n = n\, logx]$

Differentiating with respective $'x'$ on both sides

$\frac{1}{y_1} \times \frac{dy_1}{dx} = tanx \frac{d}{dx}(logx) + logx \frac{d}{dx}(tanx)$ $[\because (uv)' = uv' + vu']$

$\Rightarrow \frac{dy_1}{dx} = y_1 \left[\frac{tanx}{x} + sec^2xlogx\right]$ $\left[\because \frac{d}{dx}(logx) = \frac{1}{x}\right] \left[\because \frac{d}{dx}(tanx) = sec^2x\right]$

$\frac{dy_1}{dx} = x^{tanx} \left[\frac{tanx}{x} + sec^2x.logx\right]$ ------ (2)

$y_2 = (sinx)^{cosx}$

Applying log on both sides, $y_2 = log(sinx)^{cosx}$

$\Rightarrow logy_2 = cosx.\log(sinx)$

Differentiating with respective $'x'$ on both sides

$\frac{1}{y_2} . \frac{dy_2}{dx} = cosx \frac{d}{dx}[log(sinx)] + \log(sinx) \frac{d}{dx}(cosx)$ $[\because (uv)' = uv' + vu']$

$\Rightarrow \frac{1}{y_2} . \frac{dy_2}{dx} = cosx \left[\frac{1}{sinx} \times cosx\right] - sinx\, log(sinx)$

$\Rightarrow \frac{dy_2}{dx} = y_2[cosx\, cotx - sinx \log(sinx)]$

$\Rightarrow \frac{dy_2}{dx} = (sinx)^{cosx}[cosx.cotx - sinx \log(sinx)]$ ---------(3)

Substituting (2) & (3) in equation (1) we get

$\frac{dy}{dx} = x^{tanx} \left[\frac{tanx}{x} + sec^2x.x\, logx\right] + (sinx)^{cosx} [cosx.cotx - sinx \log(sinx)]$

4. Find the derivative of $(sinx)^{logx} + x^{sinx}$ with respective of x.

Sol: Let $y = (sinx)^{logx} + x^{sinx}$

Now let $y_1 = (sinx)^{logx}$, $y_2 = x^{sinx}$ then

$y = y_1 + y_2$

$\frac{dy}{dx} = \frac{dy_1}{dx} + \frac{dy_2}{dx}$ --------- (1)

Now$y_1 = (sinx)^{logx}$

Taking log on both sides, $logy_1 = log(sinx)^{logx}$

$\Rightarrow logy_1 = logx \log(sinx) \quad [\because\ logx^n = n\ logx]$

Differentiating both sides with respect to x then

$\frac{d}{dx}(logy_1) = \frac{d}{dx}(\log x \log(sinx))$

$\Rightarrow \frac{d}{y_1}\frac{dy_1}{dx} = logx\ \frac{d}{dx}(\log(sinx)) + \log(sinx)\frac{d}{dx}(logx) \qquad [\because (uv)' = uv' + vu']$

$\Rightarrow \frac{1}{y_1}\frac{dy_1}{dx} = logx\frac{1}{sinx}cosx + \log(sinx)\frac{1}{x} \qquad \left[\because\ \frac{d}{dx}\log(x) = \frac{1}{x}\right]\left[\because\ \frac{d}{dx}(sinx) = cosx\right]$

$\Rightarrow \frac{dy_1}{dx} = y_1\left[logx\ cotx + \frac{\log(sinx)}{x}\right] \qquad \left[\because\ \frac{cosx}{sinx} = cotx\right]$

$\Rightarrow \frac{dy_1}{dx} = (sinx)^{logx}\left[cotx\ logx + \frac{\log(sinx)}{x}\right]$ ----------- (2)

Taking log on both sides, $logy_2 = logx^{sinx} \Rightarrow logy_2 = sinx\ logx$

Differentiating both sides with respect to x then $\frac{d}{dx}(logy_2) = \frac{d}{dx}(sinx.logx)$

$\Rightarrow \frac{1}{y_2}\frac{dy_2}{dx} = sinx\frac{d}{dx}(logx) + logx\frac{d}{dx}(sinx)$

$\Rightarrow \frac{dy_2}{dx} = y_2\left[sinx.\frac{1}{x} + logx.cosx\right]$

$\Rightarrow \frac{dy_2}{dx} = (x)^{sinx}\left[\frac{sinx}{x} + logx.cosx\right]$ ----------- (3)

Substitute (2) and (3) in (1)

$\frac{dy}{dx} = (sinx)^{logx}\left[cotx\ logx + \frac{1}{x}\log(sinx)\right] + x^{sinx}\left[\frac{sinx}{x} + logx\ cosx\right]$

5. If $x^y + y^x = a^b$ then show that $\frac{dy}{dx} = -\left[\frac{yx^{y-1}+y^x logy}{x^y logx+xy^{x-1}}\right]$.

Sol: Let $y_1 = x^y$ and $y_2 = y^x$

Then given equation is $y_1 + y_2 = a^b$

Differentiating with respect to x we get

$\frac{dy_1}{dx}+\frac{dy_2}{dx}=0$ ---------- (1)

Now $y_1 = x^y$

Applying log on both sides $logy_1 = logx^y$

$$logy_1 = y\,logx \qquad [\because\ logx^n = n\,logx]$$

Differentiating with respect to x

$\Rightarrow \frac{1}{y_1}\frac{dy_1}{dx} = \frac{d}{dx}[y\,logx] = \frac{1}{y_1}\frac{dy_1}{dx} = y\frac{d}{dx}(logx) + logx \times \frac{dy}{dx}$ $\qquad [\because (uv)' = uv' + vu']$

$\Rightarrow \frac{1}{y_1}\frac{dy_1}{dx} = y\frac{1}{x} + logx\frac{dy}{dx}$ $\qquad \left[\because\ \frac{d}{dx}\log(x) = \frac{1}{x}\right]$

$\Rightarrow \frac{dy_1}{dx} = y_1\left[\frac{y}{x} + logx\ \frac{dy}{dx}\right] \Rightarrow \frac{dy_1}{dx} = x^y\left[\frac{y}{x} + logx\ \frac{dy}{dx}\right]$

$\Rightarrow \frac{dy_1}{dx} = x^{y-1}.y + x^y.logx\ \frac{dy}{dx}$ ----------- (2) $\qquad \left[\because\ \frac{a^m}{a^n} = a^{m-n}\right]$

Also $y_2 = y^x$

Applying log on both sides, $logy_2 = logy^x \Rightarrow logy_2 = x\,logy$ $\qquad [\because\ logx^n = n\,logx]$

Differentiating both sides with respect to x

$\Rightarrow \frac{1}{y_2}\frac{dy_2}{dx} = \frac{d}{dx}[x\,logy] \Rightarrow \frac{1}{y_2}\frac{dy_2}{dx} = x\frac{d}{dx}(logy) + logy \times \frac{dy}{dx}(x)$

$\Rightarrow \frac{1}{y_2}\frac{dy_2}{dx} = x\frac{1}{y}\frac{dy}{dx} + logy.1 \Rightarrow \frac{dy_2}{dx} = y_2\left[\frac{x}{y}\frac{dy}{dx} + logy\right]$

$\Rightarrow \frac{dy_2}{dx} = y^x\left[\frac{x}{y}\frac{dy}{dx} + logy\ \right] \ \Rightarrow \frac{dy_2}{dx} = y^{x-1}.x\frac{dy}{dx} + y^x.logy$ ---------(3)

Substitute (2) and (3) in (1)

$x^{y-1}.y + x^y.logx\ \frac{dy}{dx} + y^{x-1}.x\frac{dy}{dx} + y^x.logy\ = 0$

$\Rightarrow \frac{dy}{dx}(x^y logx + x^{y-1}x) = -(y^x logy + x^{y-1}y)$

$\Rightarrow \frac{dy}{dx} = \frac{-(y^x logy + x^{y-1}y)}{(x^y logx + x^{y-1}x)}$.

6. If $y = x\sqrt{a^2+x^2} + a^2\log\left(x+\sqrt{a^2+x^2}\right)$ then prove that $\frac{dy}{dx} = 2\sqrt{a^2+x^2}$.

Sol: Given $y = x\sqrt{a^2+x^2} + a^2\log\left(x+\sqrt{a^2+x^2}\right)$

Differentiating with respect to x on both sides

$\frac{dy}{dx} = \sqrt{a^2+x^2}\frac{d(x)}{dx} + x\frac{d}{dx}\left(\sqrt{a^2+x^2}\right) + a^2.\frac{d}{dx}\log\left(x+\sqrt{a^2+x^2}\right)$ $[\because (uv)' = uv' + vu']$

$\Rightarrow \frac{dy}{dx} = \sqrt{a^2+x^2} + x.\frac{2x}{2\sqrt{a^2+x^2}} + a^2.\frac{1}{x+\sqrt{a^2+x^2}}\frac{d}{dx}\left(log\left(x+\sqrt{a^2+x^2}\right)\right)$

$\left[\because \frac{d}{dx}(x) = 1\right]\left[\because \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}\right]\left[\because \frac{d}{dx}(x^n) = nx^{n-1}\right]\left[\because \frac{d}{dx}(logx) = \frac{1}{x}\right]\left[\because \frac{d}{dx}(k) = 0\right]$

$\Rightarrow \frac{dy}{dx} = \sqrt{a^2+x^2} + \frac{x^2}{\sqrt{a^2+x^2}} + \frac{a^2}{x+\sqrt{a^2+x^2}}\left[1 + \frac{1}{2\sqrt{a^2+x^2}}.\frac{d}{dx}(a^e + x^2)\right]$

$\Rightarrow \frac{dy}{dx} = \sqrt{a^2+x^2} + \frac{x^2}{\sqrt{a^2+x^2}} + \frac{a^2}{x+\sqrt{a^2+x^2}}\left[\frac{1}{1} + \frac{1}{2\sqrt{a^2+x^2}}(2x)\right]$

$\Rightarrow \frac{dy}{dx} = \sqrt{a^2+x^2} + \frac{x^2}{\sqrt{a^2+x^2}} + \frac{a^2}{x+\sqrt{a^2+x^2}}\left[\frac{\sqrt{a^2+x^2}+x}{\sqrt{a^2+x^2}}\right]$

$\Rightarrow \frac{dy}{dx} = \sqrt{a^2+x^2} + \frac{x^2}{\sqrt{a^2+x^2}} + \frac{a^2}{\sqrt{a^2+x^2}} = \sqrt{a^2+x^2} + \frac{a^2+x^2}{\sqrt{a^2+x^2}}$

$\Rightarrow \frac{dy}{dx} = \sqrt{a^2+x^2} + \frac{a^2+x^2}{\sqrt{a^2+x^2}}$ $[\because a = \sqrt{a}\sqrt{a}]$

$\Rightarrow \frac{dy}{dx} = \sqrt{a^2+x^2} + \sqrt{a^2+x^2} = 2\sqrt{a^2+x^2}$

7. If $x^y = y^x$ then show that $\frac{dy}{dx} = \frac{y(xlogy-y)}{x(y\,logx-x)}$

Sol: Given $x^y = y^x$

Applying log on both sides, $logx^y = logy^x \Rightarrow y\,logx = x\,logy$

Differentiating with respect to x on both sides

$y\frac{d}{dx}(logx) + logx\frac{d}{dx}(y) = x\frac{d}{dx}(logy) + logy\frac{d}{dx}(x)$ $[\because (uv)' = uv' + vu']$

$\Rightarrow y\frac{1}{x} + logx.\frac{dy}{dx} = x.\frac{1}{y}.\frac{dy}{dx} + logy.1$

$\Rightarrow \frac{y}{x} + logx.\frac{dy}{dx} = \frac{x}{y}.\frac{dy}{dx} + logy \Rightarrow \frac{dy}{dx}\left[\frac{logx}{1} - \frac{x}{y}\right] = \frac{logx}{1} - \frac{y}{x}$

$\Rightarrow \frac{dy}{dx}\left[\frac{y\,logx-x}{y}\right] = \frac{x\,logy-y}{x} \Rightarrow \frac{dy}{dx} = \frac{y(xlogy-y)}{x(y\,logx-x)}$

8. If $x^{logy} = logx$ then show that $\frac{dy}{dx} = \frac{y}{x}\frac{[1-logx\,logy]}{log^2x}$.

Sol: Given $x^{logy} = logx$

Take 'log' on both sides, $log(x^{logy}) = \log(logx)$

$\Rightarrow logy.logx = \log(logx)$

Differentiating with respect to x we get

$$logy.\frac{d}{dx}(logx) + logx\frac{d}{dx}(\log y) = \frac{d}{dx}(\log(logx))$$

$$\Rightarrow \frac{logx}{y}\frac{dy}{dx} = \frac{1}{x\, logx} - \frac{logy}{x}$$

$$\Rightarrow \frac{logx}{y}\frac{dy}{dx} = \frac{1-logx\, logy}{x\, logx} \Rightarrow \frac{dy}{dx} = \frac{y}{x}\frac{(1-logx\, logy)}{log^2 x}$$

7 Marks LEVEL - 2

1. If $a > b > 0$ and $0 < x < \pi$; $f(x) = (a^2 - b^2)^{-1/2}.cos^{-1}\left(\frac{a\, cosx+b}{a+b\, cosx}\right)$ then $f'(x) = (a + b\, cosx)^{-1}$

Sol: Let $f(x) = (a^2 - b^2)^{-1/2}.cos^{-1}\left(\frac{a\, cosx+b}{a+b\, cosx}\right)$

Differentiating with respect to x we get

$$f'(x) = (a^2 - b^2)^{-1/2}\left[cos^{-1}\left(\frac{b+ acosx}{a+b\, cosx}\right)\right]'$$

$$= (a^2 - b^2)^{-1/2}\left[\frac{-1}{\sqrt{1-\left(\frac{b+a\, cosx}{a+bcosx}\right)^2}}\left[\frac{b+a\, cosx}{a+b\, cosx}\right]'\right] \qquad \left[\because \frac{d}{dx}(cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}\right]$$

$$= (a^2 - b^2)^{-1/2}\left[\frac{-1}{\sqrt{1-\frac{(b+a\, cosx)^2}{(a+bcosx)^2}}} \times \left[\frac{(a+bcosx)(b+acosx)'-(b+acosx)(a+bcosx)'}{(a+bcosx)^2}\right]\right]$$

$$= (a^2 - b^2)^{-1/2}\left[\frac{(a+bcosx)}{\sqrt{(a+bcosx)^2-(b+acosx)^2}} \times \left[\frac{(b+acosx)(-bsinx)-(a+bcosx)(-asinx)}{(a+bcosx)^2}\right]\right]$$

$$= (a^2 - b^2)^{-1/2}\left[\frac{(a+bcosx)}{\sqrt{a^2+b^2cos^2x+2abcosx-b^2-a^2cos^2x-2abcosx}} \times \left[\frac{-b^2sinx-absinxcosx+a^2sinx+absinxcosx}{(a+bcosx)^2}\right]\right]$$

$$= (a^2 - b^2)^{-1/2}\left[\frac{(a^2-b^2)sinx}{(a+bcosx)\sqrt{(a^2-b^2)(1-cos^2x)}}\right]$$

$$= (a^2 - b^2)^{-1/2}\left[\frac{\sqrt{a^2-b^2}\sqrt{a^2-b^2}\,sinx}{(a+bcosx)\,\sqrt{(a^2-b^2)}\,sinx}\right]$$

$$= \frac{(a^2-b^2)^{-1/2}(a^2-b^2)^{1/2}}{(a+bcosx)} = \frac{1}{(a+bcosx)^2}$$

$$f'(x) = (a + bcosx)^{-1}$$

2. If $f(x) = sin^{-1}\sqrt{\frac{x-\beta}{\alpha-\beta}}$ and $g(x) = tan^{-1}\sqrt{\frac{x-\beta}{\alpha-x}}$ then show that $f'(x) = g'(x)$ $(\beta < x < \alpha)$

Sol: Given $f(x) = sin^{-1}\sqrt{\frac{x-\beta}{\alpha-\beta}}$

Differentiating with respect to x on both sides

$$f'(x) = \frac{1}{\sqrt{1-\left[\sqrt{\frac{x-\beta}{\alpha-\beta}}\right]^2}}\frac{d}{dx}\left[\sqrt{\frac{x-\beta}{\alpha-\beta}}\right]$$

$$\Rightarrow f'(x) = \frac{1}{\sqrt{\frac{1}{1}-\left[\sqrt{\frac{x-\beta}{\alpha-\beta}}\right]^2}} \times \frac{1}{2\sqrt{\frac{x-\beta}{\alpha-\beta}}}\,\frac{d}{dx}\left[\frac{x-\beta}{\alpha-\beta}\right]$$

$$\Rightarrow f'(x) = \frac{1}{\sqrt{\frac{\alpha-\beta-x+\beta}{\alpha-\beta}}} \times \frac{\sqrt{\alpha-\beta}}{2\sqrt{x-\beta}} \times \frac{1}{\alpha-\beta}$$

$$\Rightarrow f'(x) = \frac{\sqrt{\alpha-\beta}}{\sqrt{\alpha-x}} \times \frac{\sqrt{\alpha-\beta}}{2\sqrt{x-\beta}} \times \frac{1}{\alpha-\beta} \;\Rightarrow\; f'(x) = \frac{1}{2\sqrt{(\alpha-x)(x-\beta)}} \text{ --------- (1)}$$

$$g(x) = tan^{-1}\left(\sqrt{\frac{x-\beta}{\alpha-x}}\right)$$

Differentiating with respect to x on both sides

$$g'(x) = \frac{1}{\sqrt{1+\left[\sqrt{\frac{x-\beta}{\alpha-x}}\right]^2}}\frac{d}{dx}\left[\sqrt{\frac{x-\beta}{\alpha-x}}\right]$$

$$\Rightarrow g'(x) = \frac{1}{\frac{1}{1}+\frac{x-\beta}{\alpha-x}} \times \frac{1}{2\sqrt{\frac{x-\beta}{\alpha-x}}}\,\frac{d}{dx}\left[\frac{x-\beta}{\alpha-x}\right]$$

$$\Rightarrow g'(x) = \frac{1}{\frac{\alpha-x+x-\beta}{\alpha-x}} \times \frac{\sqrt{\alpha-x}}{2\sqrt{x-\beta}}\left[\frac{(\alpha-x)\frac{d}{dx}(x-\beta)-(x-\beta)\frac{d}{dx}(\alpha-x)}{(\alpha-x)^2}\right]$$

$$\Rightarrow g'(x) = \frac{(\alpha-x)\sqrt{\alpha-x}}{(\alpha-\beta)\sqrt{x-\beta}}\left[\frac{\alpha-x+x-\beta}{(\alpha-x)^2}\right]$$

$\Rightarrow g'(x) = \frac{(\alpha-x)\sqrt{\alpha-x}}{(\alpha-\beta)\sqrt{x-\beta}} \times \frac{(\alpha-\beta)}{(\alpha-x)^2} \Rightarrow g'(x) = \frac{\sqrt{\alpha-x}}{\sqrt[2]{x-\beta}\sqrt{\alpha-x}\sqrt{\alpha-x}}$

$\Rightarrow g'(x) = \frac{1}{\sqrt[2]{(\alpha-x)(x-\beta)}}$ --------- (2)

From (1) and (2), $f'(x) = g'(x)$

3. If $y = tan^{-1}\left[\frac{2x}{1-x^2}\right] + tan^{-1}\left[\frac{3x-x^3}{1-3x^2}\right] - tan^{-1}\left[\frac{4x-4x^3}{1-6x^2+x^4}\right]$ then show that $\frac{dy}{dx} = \frac{1}{1+x^2}$

Sol: Given $y = tan^{-1}\left[\frac{2x}{1-x^2}\right] + tan^{-1}\left[\frac{3x-x^3}{1-3x^2}\right] - tan^{-1}\left[\frac{4x-4x^3}{1-6x^2+x^4}\right]$

Put $x = tan\theta \Rightarrow \theta = tan^{-1}(x)$

$y = tan^{-1}\left[\frac{2\, tan\theta}{1-tan^2\theta}\right] + tan^{-1}\left[\frac{3\, tan\theta - tan^3\theta}{1-3tan^2\theta}\right] - tan^{-1}\left[\frac{4tan\theta - 4tan^3\theta}{1-6tan^2\theta+tan^4\theta}\right]$

$\Rightarrow y = tan^{-1}(tan2\theta) + tan^{-1}(tan3\theta) - tan^{-1}(tan4\theta)$

$\Rightarrow y = 2\theta + 3\theta - 4\theta = 5\theta - 4\theta$

$\Rightarrow y = \theta = tan^{-1}x$

Differentiating with respect to $'x'$ we get

$\frac{dy}{dx} = \frac{1}{1+x^2}$

4. Find the derivative of $f(x) = tan^{-1}\left(\frac{2x}{1-x^2}\right)$ with respect to $g(x) = sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Sol: Given $f(x) = tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Put $x = tan\theta \Rightarrow \theta = tan^{-1}x$

$f(x) = tan^{-1}\left(\frac{2\, tan\theta}{1-tan^2\theta}\right)$

$\Rightarrow f(x) = tan^{-1}(tan2\theta) = 2\theta$

$\Rightarrow f(x) = 2tan^{-1}x$

Differentiating with respect to $'x'$

$f'(x) = 2 \times \frac{1}{1+x^2} \Rightarrow f'(x) = \frac{2}{1+x^2}$

$g(x) = sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Put $x = tan\theta \Rightarrow \theta = tan^{-1}x$

$g(x) = \ sin^{-1}\left(\frac{2\ tan\theta}{1+tan^2\theta}\right)$

$\Rightarrow g(x) = \ sin^{-1}(sin2\theta) = 2\theta$

$\Rightarrow g(x) = 2tan^{-1}x$

Differentiating with respect to $'x'$

$g'(x) = 2 \times \frac{1}{1+x^2} \ \Rightarrow \ g'(x) = \frac{2}{1+x^2}$

Now $\frac{f'(x)}{g'(x)} = \frac{\frac{2}{1+x^2}}{\frac{1}{1+x^2}} = 1$

5. If $ax^2 + 2hxy + by^2 = 1$ then prove that $\frac{d^2y}{dx^2} = \frac{h^2-ab}{(hx+by)^3}$.

Sol: Given $ax^2 + 2hxy + by^2 = 1$

Differentiating with respect to x on both sides

$a\frac{d}{dx}(x^2) + 2h\frac{d}{dx}9x.y) + b\frac{d}{dx}(y^2) = \frac{d}{dx}(1)$

$a(2x) + 2h\left[x\frac{dy}{dx} + y\right] + 2by\ \frac{dy}{dx} = 0$

$\Rightarrow 2ax + 2hx\frac{dy}{dx} + 2hy + 2by\frac{dy}{dx} = 0$

$\Rightarrow 2(ax + by) + 2\frac{dy}{dx}(hx + by) = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-(ax+by)}{hx+by}$ ----------- (1)

Again Differentiating with respect to "x" on both sides

$\frac{d}{dx}\left[\frac{dy}{dx}\right] = -\frac{d}{dx}\left[\frac{(ax+by)}{hx+by}\right]$

$\frac{d^2y}{dx^2} = \left[\frac{(hx+by)\frac{d}{dx}(ax+by)-(ax+by)\frac{d}{dx}(hx+by)}{(hx+by)^2}\right]$

$= -\left[\frac{(hx+by)\left[a+h\frac{dy}{dx}\right]-(ax+by)\left[h+b\frac{dy}{dx}\right]}{(hx+by)^2}\right]$

$= -\frac{hax+h^2\frac{dy}{dx}\times x+aby+hby\frac{dy}{dx}-hax-bax\frac{dy}{dx}-h^2y-hby\frac{dy}{dx}}{(hx+by)^2}$

$= -\left[\frac{\frac{dy}{dx}(h^2x-bax)+y(ab-h^2)}{(hx+by)^2}\right]$

$$= -\left[\frac{\frac{-(ax+by)}{hx+by}\left(h^2-bax\right)+\frac{y(ab-h^2)}{1}}{(hx+by)^2}\right] \qquad [\because from\ eq\ (1)]$$

$$= -\frac{-(ax+by)\left(h^2x-bax\right)+y(ab-h^2)(hx+by)}{(hx+by)^2}$$

$$= -\left[\frac{h^2\left(ax^2+by^2+hxy+hxy\right)+ab\left(ax^2+by^2+hxy+hxy\right)}{(hx+by)^3}\right]$$

$$= -\left[\frac{h^2\left(ax^2+2hxy+by^2\right)-ab\left(ax^2++2hxy+by^2\right)}{(hx+by)^3}\right]$$

$$= \frac{\left(ax^2+2hxy+by^2\right)\left(h^2-ab\right)}{(hx+by)^2} \qquad [\because given\ ax^2+2hxy+by^2=1]$$

$$\therefore \qquad \frac{d^2y}{dx^2} = \frac{h^2-ab}{(hx+by)^3}$$

10. APPLICATION OF DERIVATIVES

10.1 ERRORS AND APPROXIMATIONS

Synopsis point

- If $y = f(x)$ is a differentiable function of x and
  $\Delta x \to$ change in x, then actual change in y is
  $\Delta y = f(x + \Delta x) - f(x);\ dy = f'(x)\Delta x$
- Approximations:- The approximate value of $f(x)$ in Δx neighborhood of known x is
  $f(x + \Delta x) = f(x) + f'(x)\Delta x.$
  $(\Delta x = given\ value\ of\ x - known\ value\ of\ x$
- If in $y = f(x)$, an error Δx occurs in x then
  (i) Δy is called error in y.
  (ii) $\frac{\Delta y}{y}$ is called relative error in y.
  (iii) $\frac{\Delta y}{y} \times 100$ is called percentage error in y.

LEVEL - I

1. If $y = x^2 + 3x + 6$ then find Δy and dy when $x = 10, \Delta x = 0.01$.

Sol: Given $y = x^2 + 3x + 6,\ x = 10,\ \Delta x = 0.01$

(i) $\Delta y = f(x + \Delta x) - f(x)$

$= (x + \Delta x)^2 + 3(x + \Delta x) + 6 - (x^2 + 3x + 6)$

$= x^2 + 2x\Delta x + (\Delta x)^2 + 3x + 3\Delta x + 6 - x^2 - 3x - 6$

$= 2x\Delta x + (\Delta x)^2 + 3\Delta x = \Delta x(2x + \Delta x + 3)$

$= 0.01[2(10) + 0.01 + 3]$

$= (0.01)(23.01) = 0.2301$

(ii) $dy = f'(x)\Delta x = (2x + 3)\Delta x = [2(10) + 3](0.01) = 23(0.01) = 0.23$

2. Find Δy and dy for the function $y = x^2 + x$, when $x = 10,\ \Delta x = 0.1$

Sol: Let $y = f(x) = x^2 + x,\ x = 10,\ \Delta x = 0.1$

(i) $\Delta y = f(x + \Delta x) - f(x)$

$= (x + \Delta x)^2 + (x + \Delta x) - x^2 - x$

$= x^2 + 2x\Delta x + (\Delta x)^2 + \Delta x - x^2 - x$

$= \Delta x(\Delta x + 2x + 1) = 0.01(0.1 + 2(10) + 1)$

$= (0.1)(0.1 + 21) = (0.1)(21.1) = 2.11$

(ii) $dy = f'(x)\Delta x = (2x + 1)\Delta x$

$= [2(10) + 1]0.1 = 21(0.1) = 2.1$

3. Find Δy and dy for the function $y = 1/(x + 2)$, where $x = 8,\ \Delta x = 0.02$

Sol: Let $y = f(x) = \frac{1}{x+2}$ and $x = 8,\ \Delta x = 0.02$

(i) $\Delta y = f(x + \Delta x) - f(x) = \frac{1}{(x+\Delta x)+2} - \frac{1}{x+2}$

$= \frac{1}{8+0.02+2} - \frac{1}{8+2} = \frac{1}{10.02} - \frac{1}{10}$

4. Find the approximate value of $\sqrt{82}$.

Sol: $\sqrt{82} = \sqrt{81 + 1}$

∴ Known value $x = 81$ and $\Delta x = 1$

Let $f(x) = \sqrt{x} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}}$

∴ approximate value is given by

$f(x + \Delta x) = [f(x) + f'(x)\Delta x]_{at\ \ known\ x}$

$\therefore \sqrt{82} = \left(\sqrt{x} + \frac{1}{2\sqrt{x}}\Delta x\right)_{at\ known\ x=81}$

$= \sqrt{81} + \frac{1}{2\sqrt{81}}(1)$

$= 9 + \frac{1}{2(9)} = 9 + \frac{1}{18} = 9 + 0.0555 = 9.0555$

5. Find the approximate value of $\sqrt[3]{65}$.

Sol: $\sqrt[3]{65} = \sqrt[3]{64 + 1}$

Known value $x = 64$ and $\Delta x = 1$

Let $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}} \Rightarrow f'(x) = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}}$

∴ approximate value in given by

$f(x + \Delta x) = [f(x) + f'(x)\Delta x]_{at\ known\ x}$

∴ $\sqrt[3]{65}$ = $\sqrt[3]{x} + \frac{1}{3x^{2/3}}\Delta x = \sqrt[3]{64} + \frac{1}{3(64)^{\frac{2}{3}}}(1)$

$= 4 + \frac{1}{3(64^3)^{\frac{2}{3}}}(1) = 4 + \frac{1}{3(4^2)} = 4 + \frac{1}{3(16)}$

$= 4 + \frac{1}{48} = \frac{192+1}{48} = \frac{193}{48} = 4.0208$

6. If the increase in the side of a square is 4% then find the approximate percentage of increase in the area of square.

Sol: let x is the side of the square

Given $\frac{dx}{x} \times 100 = 4$

Area of the square A = x^2

$\Rightarrow logA = logx^2 = 2logx$

$\Rightarrow \frac{1}{A}dA = 2\frac{1}{2}\ dx$

$\Rightarrow \frac{dA}{A} \times 100 = 2\frac{dx}{x} \times 100$ = 2(4) = 8

7. If the increase in the side of a square is 2% then find the approximate percentage of increase in the area of the square.

Sol: let x is the side of the square

Given $\frac{dx}{x} \times 100 = 2$

Area of the square A = x^2

$\Rightarrow logA = logx^2 = 2logx$

$\therefore logA = 2\ logx \quad \Rightarrow \frac{1}{A}dA = 2\frac{1}{2}\ dx$

$\Rightarrow \frac{dA}{A} \times 100 = 2\frac{dx}{x} \times 100$ = 2(2) = 4

8. The side of a square is increased from 3cm to 3.01cm. find the approximate increase in the area of the square.

Sol: Let x denotes the side of the square

$x = 3$ and $\Delta x = 3.01 - 3 = 0.01$

Area of the square $A = x^2$

$\Rightarrow \Delta A = \frac{dA}{dx}\Delta x = (2x)\Delta x$

$= 2(3)(0.01) = 0.06$sq. units

9. If the radius of the sphere is increased from 7cm to 7.02cm then find the approximate increase in the volume of the sphere.

Sol: Let r = radius of the sphere

r = 7cm and Δr = $7.02 - 7 = 0.02$cm

Volume of the sphere $v = \frac{4}{3}\pi r^3$

$\Rightarrow \Delta v = \frac{dv}{dr}\Delta r$

$= \frac{4}{3}\pi(3r^2)\Delta r = 4\pi r^2 \Delta r$

$= \frac{4(22)(7)(7)(0.02)}{7} = 12.32cm^3$

LEVEL - 2

1. $y = cosx$ then find Δy and dy when $x = 60^o$ and $\Delta x = 1^o = 0.0174$.

Sol: $\Delta y = f(x + \Delta x) - f(x)$

$= cos(60^o + 1^o) - cos60^o = cos61^o - cos60^o$ $[\because cos1^o = 0.4848]$

$\therefore \Delta y = 0.4848 - \frac{1}{2} = 0.4848 - 0.5 = -0.0152$

(ii) $dy = f'(x)\Delta x = (-sinx)\Delta x = (-sin60^o)(1^o) = \left(-\frac{\sqrt{3}}{2}\right)(0.0174)$

$= (-0.8660)(0.0174) = -0.01506$

2. Find the approximate value of $\sqrt[3]{999}$.

Sol: $\sqrt[3]{999} = \sqrt[3]{1000-1}$

Known value $x = 1000$ and $\Delta x = -1$

Let $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}} \Rightarrow f'(x) = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}}$

∴ approximate value in given by

$f(x+\Delta x) = [f(x) + f'(x)\Delta x]_{at\ known\ x}$

$\therefore \sqrt[3]{999} = \sqrt[3]{x} + \frac{1}{3x^{2/3}}\Delta x = \sqrt[3]{1000} + \frac{1}{3(1000)^{\frac{2}{3}}}(-1)$

$= 10 + \frac{1}{3(10^3)^{\frac{2}{3}}}(-1) = 10 - \frac{1}{3(10^2)} = 10 - \frac{1}{3(100)}$

$= 10 - \frac{1}{300} = 10 - 0.0033 = 9.9969$

3. Find the approximate value of $\sqrt[3]{7.8}$.

Sol: $\sqrt[3]{7.8} = \sqrt[3]{8-0.2}$

Known value $x = 8$ and $\Delta x = -0.2$

Let $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}} \Rightarrow f'(x) = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}}$

∴ approximate value in given by

$f(x+\Delta x) = [f(x) + f'(x)\Delta x]_{at\ known\ x}$

$\therefore \sqrt[3]{65} = \sqrt[3]{x} + \frac{1}{3x^{2/3}}\Delta x = \sqrt[3]{8} + \frac{1}{3(8)^{\frac{2}{3}}}(-0.2)$

$= 2 - \frac{1}{3(2^3)^{\frac{2}{3}}}(0.2) = 2 - \frac{0.2}{3(2^2)} = 2 - \frac{0.2}{12}$

$= 2 - 0.0166 = 1.9834$

4. Find the approximate value of $sin62^o$.

Sol: $sin62^o = sin(60^o + 2^o)$ known value $x = 60^o$ and

$\Delta x = 2^o - 2(0.0174)^o$

Let $f(x) = sinx \Rightarrow f'(x) = cosx$

∴ approximate value is given by

$f(x+\Delta x) = [f(x) + f'(x)\Delta x]_{at\ known\ x}$

$sin62^o = (sinx + cosx\ \Delta x)$ at $x = 60^o$

$= sin60^o + cos60^o \times 2^o$

$= \frac{\sqrt{3}}{2} + \frac{1}{2} \times 2^o = \frac{\sqrt{3}}{2} + \frac{1}{2} \times 2 \times 0.0174$

$= 0.8660 + 0.0174 = 0.8834$

5. The time $'t'$ of a complete oscillation of a simple pendulum of length l is given by $t = 2\pi\sqrt{\frac{l}{g}}$ where g is gravitational constant. Find the approximate percentage of error in $'t'$ when the percentage of error in l is 1%.

Sol: Given that $\frac{dl}{l} \times 100 = 1$ Given equation is $t = 2\pi\sqrt{\frac{l}{g}}$

$= \frac{2\pi}{\sqrt{g}}\sqrt{l}$ [∵ g is constant]

$\Rightarrow logt = log\frac{2\pi}{\sqrt{g}} + log\sqrt{l}$ [$\because logab = \log a + \log b$]

$\Rightarrow \log t = \log\frac{2\pi}{\sqrt{g}} + \frac{1}{2}\log l$

$\Rightarrow \frac{1}{t}\ dt = 0 + \frac{1}{2}\frac{1}{l}dl$

$\Rightarrow \frac{dt}{t} \times 100 = \frac{1}{2}\frac{1}{l}dl \times 100 = \frac{1}{2}(1) = \frac{1}{2}$

PRACTICE QUESTIONS

1. If $y = 5x^2 + 6x + 6$, item find Δy and dy when $x = 2$, $\Delta x = 0.001$ [Ans: 0.026005, 0.026]
2. Find the approximate value of $\sqrt{25.001}$. [Ans: 5..0001]
3. Find the approximate value of $\sqrt[3]{17}$. [Ans: 2.0312]
4. The diameter of a sphere in measured to 40 cm. If an error of 0.02cm is made in 14, then find approximate errors in volume and surface are of the sphere. [Ans: 1.6π sq. cm]
5. If $y = f(x) = Kx^n$ then show that the approximate relative error (or increase) in y is n times. The relative error (or increase) in x where n and K are constants.

10.2 TANGENT AND NORMAL $(1\times 4 = 4M, 1 \times 7 = 7M)$

SYNOPSIS POINTS

- Tangent: The tangent is a straight line which just touches the curve at a given point.
- NormType equation here.al: The normal is a straight line which is perpendicular to the tangent.
- If $P(x_1, y_1)$ is a point on $y = f(x)$ then slope of the tangent at P is $m = \left(\frac{dy}{dx}\right)_{P(x_1,y_1)}$.
- If $P(x_1, y_1)$ is a point of intersection of the curves $f(x), g(x)$ and θ is the angle between the two curves then $tan\theta = \frac{m_1-m_2}{1+m_1m_2}$ where $m_1 = f'(x)$ or $\frac{df}{dx}$ [∵ at (x_1, y_1)]

$m_2 = g'(x)$ (or) $\frac{dg}{dx}$ [∵ at (x_1, y_1)]

Note: $m_1 = m_2$ ⇒ the two curves touch each other at (x_1, y_1)

$m_1m_2 = -1$ ⇒ the two curves cut orthogonally.

- If $P(x_1, y_1)$ is a point on the curve $y = f(x)$ and $m = \frac{dy}{dx}$ at (x_1, y_1) then

(i) Length of the tangent to the curve at $P(x_1, y_1)$ is $\left|\frac{y_1\sqrt{1+m^2}}{m}\right|$

(ii) Length of the normal to the curve at $P(x_1, y_1)$ is $\left|y_1\sqrt{1+m^2}\right|$.

(iii) Length of the sub tangent to the curve at $P(x_1, y_1)$ is $\left|\frac{y_1}{m}\right|$.

(iv) Length of subnormal to the curve at $P(x_1, y_1)$ is $|y_1m|$

LEVEL – 1 SAQ (4M)

1. Find the equations of the tangent and normal to the curve $xy = 10$ at (2, 5).

Sol: Given curve is $xy = 10$, let P(2, 5) be the given points

Differentiating w.r.t. $'x'$ $x\frac{dy}{dx} + y = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{-y}{m}$$

(i) Slope of the tangent at P(2, 5) is

$$m_1 = \frac{dy}{dx}\bigg| = \frac{-5}{2}$$

(ii) Equation of the tangent with slope $m_1 = \frac{-5}{2}$ at P(2, 5) point is

$y - y_1 = m_1(x - x_1)$

$\Rightarrow (y-5) = \frac{-5}{2}(x-2)$

$\Rightarrow 5x + 2y - 20 = 0$

(iii) Slope of the normal is $m_2 = \frac{-1}{m_1} = \frac{2}{5}$

(iv) equation of the normal at (2, 5) with slope $m_2 = \frac{2}{5}$ is

$y - y_1 = m_2(x - x_1)$

$\Rightarrow (y-5) = \frac{2}{5}(x-2)$

$\Rightarrow 2x - 5y + 21 = 0$

2. Find the equations of the tangent and the normal to the curve $y^4 = ax^3$ at (a, a).

Sol: Given curve is $y^4 = ax^3$

Different w.r.t. we get $4y^3\frac{dy}{dx} = 3ax^2$

$\Rightarrow \frac{dy}{dx} = \frac{3ax^2}{4y^3}$

(i) Slope of the tangent at (a, a) is $m = \frac{3a(a^2)}{4(a^3)} = \frac{3}{4}$

(ii) Equation of the tangent at (a, a), $m = \frac{3}{4}$ is $y - y_1 = m(x - x_1)$

$\Rightarrow y - a = \frac{3}{4}(x - a)$

$\Rightarrow 3x - 4y + a = 0$

(iii) Slope of the normal is $\frac{-1}{m} = \frac{-1}{\frac{3}{4}} = \frac{-4}{3}$

(iv) Equation of the normal at (a, a), slope $m = \frac{-4}{m}$ is

$y - y_1 = m(x - x_1)$

$\Rightarrow y - a = \frac{-4}{3}(x - a)$

$\Rightarrow 4x + 3y - 7a = 0$

3. Find the equation of the tangent to the curve $y = 3x^2 - x^3$, where it meets the x-axis.

Sol: Equation of x-axis is $y = 0$

Given curve is $y = 3x^2 - x^3$

To find the point of intersection of the line $y = 0$ and $y = 3x^2 - x^3$

Put $y = 0$ in $y = 3x^2 - x^3$

$\Rightarrow 3x^2 - x^3 = 0$

$\Rightarrow x^2(3 - x) = 0 \quad \Rightarrow x = 0, \; x = 3$

∴ Points are P(0, 0), Q(3, 0)

(a) Equation of tangent at P(0, 0)

Slope $m = \frac{dy}{dx}\Big| = \; (6x - 3x^2)| = 0$ [at P(0, 0)]

Equation of tangent at P(0, 0) and slope m = 0 is

$y - y_1 = m(x - x_1)$

$\Rightarrow y - 0 = 0(x - 0) \Rightarrow y = 0$

(b) Equation of tangent at P(3, 0).

Slope $m = \frac{dy}{dx}\Big| = \; (6x - 3x^2)| = \; 18 - 27 = -9$ [at P(3, 0)]

Equation of tangent at P(3, 0), slope m = -9 is

$y - y_1 = m(x - x_1)$

$\Rightarrow y - 0 = -9(x - 3)$

$\Rightarrow 9x + y - 27 = 0$

4. Show that the equation of the tangent to the curve $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2 \; (a \neq 0, b \neq 0)$ at the point (a, b) is
$\frac{x}{a} + \frac{y}{b} = 2.$

Sol: The given equation is $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2 \Rightarrow \frac{1}{a^n} x^n + \frac{1}{b^n} y^n = 2$ ------ (1)

Differentiating w.r.t. $'x'$ $\frac{1}{a^n}\, n. x^{n-1} + \frac{1}{b^n} n. y^{n-1} \frac{dy}{dx} = 0$

$\Rightarrow \frac{n}{a}\left(\frac{x}{a}\right)^{n-1} + \left(\frac{n}{b}\right)\left(\frac{y}{b}\right)^{n-1}\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-n}{a}\left(\frac{x}{a}\right)^{n-1}\frac{b}{n}\left(\frac{b}{y}\right)^{n-1}$

Slope of the tangent at (a, b) is $m = \frac{dy}{dx}\Big| = -\left(\frac{n}{a}\right)\left(\frac{a}{a}\right)^{n-1}\left(\frac{b}{n}\right)\left(\frac{b}{b}\right)^{n-1}$ [at P(a, b)]

$\therefore m = \frac{-b}{a}$

equation of tangent with slope $\frac{-b}{a}$ at the point (a, b) is

$y - b = \frac{-b}{a}(x - a) \Rightarrow ay - ab = -bx + ab \Rightarrow bx + ay = 2ab$

$\Rightarrow \frac{bx}{ab} + \frac{ay}{ab} = 2 \Rightarrow \frac{x}{a} + \frac{y}{b} = 2.$

5. Show that the tangent at any point θ on the curve $x = c\ cos\theta$, $y = c\ tan\theta$ is $y\ sin\theta = x - c\ cos\theta$.

Sol: Slope of the tangent at any point θ, $(c\ sec\theta, c\ tan\theta)$ on the curve is

$m = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}(c\ tan\theta)}{\frac{d}{d\theta}(c\ sec\theta)} = \frac{c\ sec^2\theta}{c\ sec\theta\ tan\theta} = cosec\theta$

$\therefore$ The equation of the tangent with slope $cosec\theta$ at θ $(c\ sec\theta,\ c\ tan\theta)$ is

$y - c\ tan\theta = cosec\theta(x - c\ sec\theta)$

$\Rightarrow y - c\ \frac{sin\theta}{cos\theta} = \frac{1}{sin\theta}\left(x - \frac{c}{cos\theta}\right)$

$\Rightarrow \frac{y\ cos\theta - c\ sin\theta}{cos\theta} = \frac{1}{sin\theta}\left(\frac{x\ cos\theta - c}{cos\theta}\right)$

$\Rightarrow y\ sin\theta\ cos\theta - c\ sin^2\theta = x\ cos\theta - c$

$\Rightarrow x\ cos\theta - y\ sin\theta\ cos\theta - c(1 - sin^2\theta) = 0$

$\Rightarrow x\ cos\theta - y\ sin\theta\ cos\theta - c(cos^2\theta) = 0$

$\Rightarrow cos\theta\ (x - y\ sin\theta - c\ cos\theta) = 0$

$\Rightarrow x - y\ sin\theta - c\ cos\theta = 0$

$\Rightarrow y\ sin\theta = x - cos\theta$

6. Find lengths of normal and subnormal at a point on the $y = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right)$.

Sol: Given that $y = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right)$

$$= a\left(\frac{e^{x/a}+e^{-x/a}}{2}\right) = a\,cosh\left(\frac{x}{a}\right)$$

$$\Rightarrow \frac{dy}{dx} = a\,sinh\left(\frac{x}{a}\right).\frac{1}{a} = sinh\left(\frac{x}{a}\right)$$

(i) length of normal = $\left|y.\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right|$

$$= \left|a\,cosh\left(\frac{x}{a}\right).\sqrt{1+sinh^2\left(\frac{x}{a}\right)}\right|$$

$$= \left|a\,cosh\left(\frac{x}{a}\right).cosh\left(\frac{x}{a}\right)\right|$$

$$= \left|a\,cosh^2\frac{x}{a}\right|$$

(ii) Length of the subnormal = $\left|y\,\frac{dy}{dx}\right| = \left|a\,cosh\frac{x}{a}\,sinh\frac{x}{a}\right|$

$$= \frac{a}{2}\,2\,sinh\frac{x}{a}\,cosh\frac{x}{a} = \left|\frac{a}{2}\,sinh\,\frac{2x}{a}\right|$$

7. Find the value of K, so that the length of the subnormal at any point on the curve $y = a^{1-K}x^K$ is a constant.

Sol: Given curve is $y = a^{1-K}x^K$

Differentiating w.r.t. $'x'$

$$\frac{dy}{dx} = a^{1-K}\,K.x^{K-1}$$

The length of the subnormal at point P(x, y) is

$$= \left|y\,\frac{dy}{dx}\right| = |y\,a^{1-K}\,K.x^{K-1}|$$

$$= |a^{1-K}.x^K\,a^{1-K}(K)x^{K-1}|$$

$$= |Ka^{2-2K}.x^{2K-1}|$$

is ‘a’ constant is $2K - 1 = 0 \Rightarrow K = \frac{1}{2}$

PRACTICE QUESTIONS (4 Marks)

1. Find thee equations of tangent and the normal to the curve $y = x^3 + 4x^2$ at (-1, 3).
2. Show that the curves $x^2 + y^2 = 2$, $3x^2 + y^2 = 4x$ have a common tangent at the point (1, 1).
3. Find the equation of tangent and normal to the curve $x = cost, y = sint$ at $t = \frac{\pi}{4}$.
4. Show that the length of subnormal at any point on the curve $xy = a^2$ varies as the cube of the ordinate of the point.

LAQ (7 Marks)

1. Show that the tangent at $P(x_1, y_1)$ on the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is $xx_1^{-\frac{1}{2}} + yy_1^{-\frac{1}{2}} = a^{\frac{1}{2}}$.

Sol: Given curve is $\sqrt{x} + \sqrt{y} = \sqrt{a}$ -------(1)

Let $P(x_1, y_1)$ be a point on the curve (1)

Differentiating (1) w.r.t. $'x'$ $\Rightarrow \sqrt{x_1} + \sqrt{y_1} = \sqrt{a}$ ------- (2)

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{1}{2\sqrt{y}}\frac{dy}{dx} = -\frac{1}{2\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2\sqrt{y}}{2\sqrt{x}} = \frac{\sqrt{y}}{\sqrt{x}}$$

Slope m at $P(x_1, y_1)$ is $\left.\frac{dy}{dx}\right| = \frac{\sqrt{y_1}}{\sqrt{x_1}}$

equation of the tangent at $P(x_1, y_1)$ m in slope $m = \frac{\sqrt{y_1}}{\sqrt{x_1}}$. is

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow y - y_1 = \frac{-\sqrt{y_1}}{\sqrt{x_1}}(x - x_1)$$

$$\Rightarrow \frac{y - y_1}{\sqrt{y_1}} = \frac{-(x - x_1)}{\sqrt{x_1}} \Rightarrow \frac{y}{\sqrt{y_1}} - \frac{y_1}{\sqrt{y_1}} = \frac{-x}{\sqrt{x_1}} + \frac{x_1}{\sqrt{x_1}}$$

$$\Rightarrow \frac{x}{\sqrt{x_1}} + \frac{y}{\sqrt{y_1}} = \frac{x_1}{\sqrt{x_1}} + \frac{y_1}{\sqrt{y_1}}$$

$$\Rightarrow x\left(x_1^{-1/2}\right) + y\left(y_1^{-1/2}\right) = \sqrt{x_1} + \sqrt{y_1} = \sqrt{a} \qquad [\because \text{from } (2)]$$

$\therefore$ locus of $P(x_1, y_1)$ is $\sqrt{x} + \sqrt{y} = \sqrt{a}$

2. If the tangent at a point on the curve $x^{2/3} + y^{2/3} = a^{2/3}$ intersects the coordinate axes in A, B then show that the length AB is constant.

Sol: Given curve is $x^{2/3} + y^{2/3} = a^{2/3}$ ------ (1)

Let $P(a\, cos^3\theta,\ a\, sin^3\theta)$ any point on the curve (1)

$\therefore x = a\, cos^3\theta \Rightarrow \frac{dx}{d\theta} = 3a\, cos^2\theta(-sin\theta)$ Differentiating w.r.t. θ

$y = asin^3\theta \Rightarrow \frac{dy}{d\theta} = 3a\, sin^2\theta\, cos\theta$

(i) Slope of the tangent 'm'at $P(\theta)$ is

$\frac{dy}{d\theta} = \frac{dy/d\theta}{dx/d\theta} = -\frac{3a\, sin^2\theta\, cos\theta}{3a\, cos^2\theta\, sin\theta} = \frac{-sin\theta}{cos\theta}$

(ii) equation tangent at $P(\theta)$ is

$y = y_1 = m(x - x_1)$

$\Rightarrow y - asin^3\theta = \frac{-sin\theta}{cos\theta}(x - a\, cos^3\theta)$

$\Rightarrow (\, y - asin^3\theta)cos\theta = -sin\theta\big((x - a\, cos^3\theta)\big)$

$\Rightarrow y\, cos\theta - asin^3\theta\, cos\theta = -xsin\theta + a\, sin\theta\, cos^3\theta$

$\Rightarrow x\, sin\theta + ycos\theta = a\, sin\theta\, cos^3\theta + a\, sin^3\theta\, cos\theta$

$\Rightarrow x\, sin\theta + ycos\theta = a\, sin\theta\, cos\theta\, (cos^2\theta + \ sin^2\theta)$

$\Rightarrow x\, sin\theta + ycos\theta = a\, sin\theta\, cos\theta$ $\because cos^2\theta + a\, sin^2\theta = 1$

$\Rightarrow \frac{x\, sin\theta}{a\, sin\theta\, cos\theta} + \frac{y\, cos\theta}{a\, sin\theta\, cos\theta} = 1$

$\Rightarrow \frac{x}{a\, cos\theta} + \frac{y}{a\, sin\theta} = 1$

$\therefore$ A($a\, cos\theta$, 0), B(0, $a\, sin\theta$)

$\therefore$ AB = $\sqrt{(a\, cos\theta - 0)^2 + (0 + a\, sin\theta)^2}$

= $\sqrt{a^2\, cos^2\theta + a^2\, sin^2\theta}$

= $\sqrt{a^2(cos^2\theta + \ sin^2\theta)}$ = $\sqrt{a^2}$ = a

$\therefore AB = a$ is a constant

3. If the tangent at any point P on the curve $x^m y^n = a^{m+n}, mn \neq 0$ meets the coordinate axes in A, B then show that AP : BP is a constant.

Sol: Given curve is $x^m y^n = a^{m+n}$ ----- (1)

Let $P\ (x_1, y_1)$ be any point on the curve

Differentiating eq(1) w.r.t. $'x'$

$$x^m \left(ny^{n-1}\frac{dy}{dx}\right) + y^n(mx^{m-1}) = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-m\, y^n x^{m-1}}{nx^m y^{n-1}} = \frac{-m}{n}\left(\frac{y}{x}\right)$$

∴ slope of the curve $m = \left.\frac{dy}{dx}\right| = \frac{-m}{n}\left(\frac{y_1}{x_1}\right)$ (at (x_1, y_1))

Equation of the tangent at $P\ (x_1, y_1)$ with slope $\frac{-m}{n}\left(\frac{y_1}{x_1}\right)$ is

$$(y - y_1) = \frac{-m}{n}\left(\frac{y_1}{x_1}\right)(x - x_1)$$

$$\Rightarrow nx_1(y - y_1) = -my_1(x - x_1)$$

$$\Rightarrow (my_1)x + (nx_1)y - x_1y_1(n + m) = 0 \text{ -------------- (2)}$$

x – intercept of the line (2) is $\frac{constant\ term}{coefficient\ of\ x} = \frac{x_1y_1(n+m)}{my_1}$

$\Rightarrow$ A(a, 0) = $\left(\frac{(m+n)x_1}{m}, 0\right)$

y – intercept of the line (2) is $\frac{constant\ term}{coefficient\ of\ y} = \frac{x_1y_1(n+m)}{nx_1}$

$\Rightarrow$ B(0, a) = $\left(0, \frac{(m+n)y_1}{n}\right)$

$P\ (x_1, y_1)$ ∴ AP : PB = $(a - x_1) : (x_1 - 0)$ = $\left(\frac{(m+n)x_1}{m} - x_1\right) : (x_1 - 0)$

$$= \frac{mx_1 + nx_1 - mx_1}{m} : x_1 = \frac{nx_1}{m} : x_1 = \frac{n}{m} : 1 = n : m$$

∴ AP : PB = $n : m$ which is a constant.

4. At any point t on the curve $x = a(t + sint)$, $y = a(1 - cost)$ find the lengths of tangent, normal, subtangent and subnormal.

Sol: Given curve is $x = a(t + sint) \Rightarrow \frac{dy}{dx} = a\ \frac{d}{dt}(t + sint)$

$\therefore \frac{dx}{dt} = a(1+cost)$ differentiating w.r.t. 't'

$y = a(1-\cos t) \Rightarrow \frac{dy}{dt} = a\ \frac{d}{dt}(1-cost)$

$\Rightarrow \frac{dy}{dt} = a\ sint$

Slope of the tangent $m = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\ sint}{a(1+cost)}$

$\therefore m = \frac{sint}{1+cost} = \frac{2sin\frac{1}{2}.cos\frac{t}{2}}{2cos^2\frac{t}{2}} = tan\frac{t}{2}$

(i) length of the tangent = $\left|\frac{y\sqrt{1+m^2}}{m}\right| = \left|\frac{a(1-\cos t)\sqrt{1+tan\frac{t}{2}}}{tan\frac{t}{2}}\right|$

$= \left|\frac{a\ 2sin^2\frac{t}{2}..\ sec\frac{t}{2}}{tan\frac{t}{2}}\right| = \left|a\ 2sin^2\frac{t}{2}.\frac{1}{cos\frac{1}{2}}.\ \frac{cos\frac{t}{2}}{sin\frac{1}{2}}\right|$

$= \left|2a\ sin\frac{t}{2}\right|$

(ii) Length of the normal = $\left|y\sqrt{1+m^2}\right| = \left|a(1-\cos t)\sqrt{1+tan\frac{t}{2}}\right|$

$= \left|a\ 2sin^2\frac{t}{2}..\ sec\frac{t}{2}\right| = \left|a\ 2sin^2\frac{t}{2}.\frac{1}{cos\frac{1}{2}}\right|$

$= \left|2a\ sin\frac{t}{2}\ cos\frac{t}{2}\right|$

(iii) Length of the subtangent = $\left|\frac{y}{m}\right|$

$= \left|\frac{a(1-\cos t)}{tan\frac{t}{2}}\right| = \left|a\ 2sin^2\frac{t}{2}.cot\frac{t}{2}\right|$

$= 2\ a\ 2sin^2\frac{t}{2}\ \frac{cos\frac{t}{2}}{sin\frac{t}{2}} = \left|a\ 2sin\frac{t}{2}.cos\frac{t}{2}\right|$

$= |a\ sint|$

(iv) Length of sub normal = $|ym| = \left|a(1-cost)tan\frac{t}{2}\right|$

$= \left|a\ \left(2sin^2\frac{t}{2}\right).tan\frac{t}{2}\right|$

$= \left|2a\ sin^2\frac{t}{2}.tan\frac{t}{2}\right|$

5. Find the angle between the curves $xy = 2$ and $x^2 + 4y = 0$.

Sol: Given curves $xy = 2$ ------- (1)

$x^2 + 4y = 0$ ----- (2)

Point of intersection of (1) & (2)

$(1) \Rightarrow y = \frac{2}{x} \quad \therefore (2) \Rightarrow x^2 + 4\left(\frac{2}{x}\right) = 0$

$\Rightarrow \frac{x^3+8}{x} = 0 \Rightarrow x^3 + 8 = 0 \Rightarrow x = -2$

$\therefore y = \frac{2}{x} = -\frac{2}{2} = -1$

$\therefore$ Point of intersection is P(-2, -1)

$xy = 2 \Rightarrow y = \frac{2}{x} \Rightarrow \frac{dy}{dx} = \frac{-2}{x^2} \therefore m_1 = \left.\frac{dy}{dx}\right| = \frac{-2}{(-2)^2} = \frac{-1}{2}$ (at -2, -1)

$x^2 + 4y = 0 \Rightarrow 2x + 4\frac{dy}{dx} = 0 \Rightarrow x + 2\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-x}{2}$

$m_2 = \left.\frac{dy}{dx}\right| = \frac{-2}{-2} = 1$ (at -2, -1)

If θ is the angle between the curves (1) & (2) at P then

$tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right| = \frac{-\frac{1}{2}-1}{1+\left(-\frac{1}{2}\times 1\right)} = \frac{-\frac{3}{2}}{\frac{1}{2}} = |-3| = 3$

$\therefore \theta = tan^{-1}3$

6. Show that curves $y^2 = 4(x+1)$, $y^2 = 36(9-x)$ intersect orthogonally.

Sol: Given curves $y^2 = 4(x+1)$ ------- (1)

$y^2 = 36(9-x)$ ----- (2)

Point of intersection of (1) & (2)

$y^2 = 4(x+1)$, $y^2 = 36(9-x)$

$4(x+1) = 36(9-x)$

$\Rightarrow x + 1 = 9(9-x)$

$\Rightarrow 10x = 80 \Rightarrow x = 8$

Put $x = 8$ in $y^2 = 4(x+1) \Rightarrow y^2 = 4(8+1) = 36$

$\therefore y = \pm 6$

$\therefore$ Points of intersection are P(8, 6), Q(8, -6)

(i) At the point P(8, 6)

$y^2 = 4(x+1) \Rightarrow 2y\frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{2}{y}$ ----- (3)

$\Rightarrow m_1 = \left(\frac{dy}{dy}\right)_{P(8,6)} = \frac{2}{6} = \frac{1}{3}$

and $y^2 = 36(9-x) \Rightarrow 2y\frac{dy}{dx} = -36 \Rightarrow \frac{dy}{dx} = \frac{-18}{y}$ ----- (4)

$\Rightarrow m_2 = \left(\frac{dy}{dy}\right)_{P(8,6)} = \frac{-18}{6} = -3$

The product of slopes of tangents at P(8, 6) is $m_1 m_2 = \left(-\frac{1}{3}\right)(3)$

$m_1 m_2 = -1$

$\Rightarrow$ the given curves intersect orthogonally at P(8, 6)

(ii) At the point Q(8, -6)

from (3) $\frac{dy}{dx} = \frac{2}{y} \Rightarrow$ $m_1 = \frac{dy}{dx}\Big| = \frac{2}{-6} = -\frac{1}{3}$ (at Q(8, -6))

from (3) $\frac{dy}{dx} = \frac{-18}{y} \Rightarrow$ $m_2 = \frac{dy}{dx}\Big| = \frac{-18}{-6} = 3$ (at Q(8, -6))

$\therefore$ The product of slopes of tangents at Q(8, -6) is $m_1 m_2 = \left(-\frac{1}{3}\right)(3)$

$m_1 m_2 = -1$

$\therefore$ The given curves intersect orthogonally at Q(8, -6)

7. Find the condition for the orthogonally of the curves $ax^2 + by^2 = 1$ and $a_1x^2 + b_1y^2 = 1$.

Sol: Given curves $ax^2 + by^2 = 1$ -------- (1)

$a_1x^2 + b_1y^2 = 1$ ------ (2)

Let $P(x_1, y_1)$ be the point of intersection of (1) & (2)

(1) $\Rightarrow ax_1^2 + by_1^2 = 1$

(2) $\Rightarrow a_1x_1^2 + b_1y_1^2 = 1$

$$\Rightarrow ax_1^2 + by_1^2 = a_1x_1^2 + b_1y_1^2$$

$$\Rightarrow x_1^2(a - a_1) = y_1^2(b_1 - b)$$

$$\Rightarrow \frac{x_1^2}{y_1^2} = \frac{b_1 - b}{a - a_1} = \frac{-(b - b_1)}{a - a_1} \text{ ----- (1)}$$

Slope of tangent (1) at $P(x_1, y_1)$ is

$$2ax + 2by\frac{dy}{dx} = 0 \Rightarrow m_1 = \left(\frac{dy}{dx}\right)_{P(x_1,y_1)} = \frac{ax_1}{by_1}$$

Slope of tangent (2) at $P(x_2, y_2)$ is

$$2a_1x + 2b_1y\frac{dy}{dx} = 0 \Rightarrow m_2 = \left(\frac{dy}{dx}\right)_{P(x_2,y_2)} = \frac{-a_1x_1}{b_1y_1}$$

Now, $m_1m_2 = -1$ (∵ curves intersect orthogonally)

$$\Rightarrow \left(\frac{-ax_1}{by_1}\right)\left(\frac{-a_1x_1}{b_1y_1}\right) = -1 \Rightarrow \frac{aa_1x_1^2}{bb_1y_1^2} = -1 \Rightarrow \frac{x_1^2}{y_1^2} = \frac{-bb_1}{aa_1} \text{ -------- (2)}$$

Equating (1) & (2)

$$\Rightarrow \frac{bb_1}{aa_1} = \frac{b - b_1}{a - a_1} \Rightarrow \frac{a - a_1}{aa_1} = \frac{b - b_1}{bb_1}$$

$$\Rightarrow \frac{a}{aa_1} - \frac{a_1}{aa_1} = \frac{b}{bb_1} - \frac{b_1}{bb_1}$$

$$\Rightarrow \frac{1}{a_1} - \frac{1}{a} = \frac{1}{b_1} - \frac{1}{b}$$

$$\Rightarrow \frac{1}{a_1} - \frac{1}{b_1} = \frac{1}{a} - \frac{1}{b}$$

Hence proved

PRACRICE QUESTIONS

1. Find the length of subtangent, sub normal at a point on the curve $x = a(cost + sint)$, $y = a(sint - cost)$.
2. Find the length of subtangent and sub normal at a point on the curve $y = b\,sin\left(\frac{x}{a}\right)$.
3. Find the angle between the curves $x + y + 2 = 0$ and $x^2 + y^2 - 10y = 0$.
4. Find the angle between the curves $2y^2 - 9x = 0$, $3x^2 + 4y = 0$.
5. Find the angle between the curves $y^2 = 4x$ and $x^2 + y^2 = 5$.
6. Find the angle between the curves $y^2 = 8x$ and $4x^2 + y^2 = 32$.

10.3 RATE MEASURE (4M)

SYNOPSIS POINTS

- Rate of change: The rate of change of a given function $y = f(x)$ is defined as $\frac{dy}{dx} = f'(x)$.

  If x and y are varying to another variable $'t'$ is if

  $x = f(t)$, $y = g(t)$ then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ $\left(\frac{dx}{dt} \neq 0\right)$

- Increasing and decreasing Functions:

  $y = f(x)$ is a function continuous in [a, b] and differentiable on the open interval (a, b) then

  (i) f is increasing in [a, b], if $f'(x) > 0 \; \forall \; x \in (a, b)$

  (ii) f is decreasing in [a, b], if $f'(x) < 0 \; \forall \; x \in (a, b)$

  (iii) f is constant in [a, b], if $f'(x) = 0 \; \forall \; x \in (a, b)$

- Local Maxima:

  Let $f(x)$ be a differentiable function in a given interval I, $a \in I$, $f'(x)$, $f''(x)$ exist at $'a'$ and if

  (i) $f'(a) = 0$, $f''(a) < 0$ then $f(a)$ is a local maxima.

  (ii) $f'(a) = 0$, $f''(a) > 0$ then $f(a)$ is a local minima.

- Stationary Point:

  A differentiable function f is said to be stationary at $x = a$ if $f'(a) = 0$, $f(a)$ is stationary value and $(a, f(a))$ in the stationary point of $f(x)$at $x = 0$.

- Velocity and Acceleration:

  If $s = f(t)$ denotes the distance travelled by a body in time 't' then (i) Velocity of the body at time 't' is $V = \frac{ds}{dt}$. (ii) Acceleration of the body at time 't' is $a = \frac{d^2s}{dt^2}$.

SAQ (4 Marks)

1. A particle moving along a straight line has the relation $s = t^3 + 2t + 3$, connecting the distance 's' describe by the particle in time 't'. Find the velocity and acceleration of the particle at t = 4 sec.

Sol: Given relation $s = t^3 + 2t + 3$

Differentiating with respect to $'x'$ $v = \frac{ds}{dt} = 3t^2 + 2$ [$\because$ velocity $v = \frac{ds}{dt}$]

Differentiating with respect to $'x'$ $a = \frac{d^2s}{dt^2} = \frac{dv}{dt} = 6t$ (a = acceleration)

At t = 4

(i) Velocity $(v) = 3(4)^2 + 2 = 3(16) + 2 = 50$ Units/Sec

(ii) Acceleration $a = 6(4) = 24$ units/sec^2

2. A particle is moving in a straight line so that after 't' seconds its distance s(in cms) from a fixed point on the line is given by $s = f(t) = 8t + t^3$. Find (i) The Velocity at time t = 2 sec (ii) The initial velocity (iii) acceleration at t = 2sec.

Sol: Given relation is $s = f(t) = 8t + t^3$ ------ (1)

Velocity (v)= $\frac{ds}{dt} = 8 + 3t^2$ -------- (2)

Acceleration $(a) = \frac{dv}{dt} = 6t$ -------- (3)

(i) from (2), the velocity at $t = 2$ is 8 + 3(4) = 20 cm/sec.

(ii) from (2), the initial velocity at $t = 0$ is 8 + 3(0) = 8 cm/sec.

(iii) from (3), the acceleration at $t = 2$ is 6(2) = 12 ccm/sec^2

3. The distance-time formula for the motion of a particle along a straight line is $s = t^3 - 9t^2 + 24t - 18$. Find when and where velocity is zero.

Sol: Given that $s = t^3 - 9t^2 + 24t - 18$

$\Rightarrow$ Velocity $(v) = \frac{ds}{dt} = 3t^2 - 18t + 24 = 3(t^2 - 6t + 8)$

$= 3(t-2)(t-4)$

If velocity is zero then $(t-2)(t-4) = 0 \Rightarrow t$ = 2 or 4

∴ The velocity becomes zero after 2 seconds and 4 seconds.

If $t = 2$ then $s = t^3 - 9t^2 + 24t - 18 = 2^3 - 9(2)^2 + 24(2) - 18$

$= 8 - 36 + 48 - 18 = 56 - 54 = 2$

If $t = 4$ then $s = t^3 - 9t^2 + 24t - 18 = 4^3 - 9(4)^2 + 24(4) - 18$

$= 64 - 144 + 96 - 18 = 160 - 162 = -2$

∴ The particle is at a distance of 2 units on either side of the starting point.

4. A particle moving olong a line according $s = f(t) = 4t^3 - 3t^2 + 5t - 1$ where s is measured in meters and 't'is measured in seconds. Find the velocity and acceleration at time 't'. at what time the acceleration is zero.

Sol: Given that $s = f(t) = 4t^3 - 3t^2 + 5t - 1$

The velocity at time 't'is $(v) = \frac{ds}{dt} = 12t^2 - 6t + 5$

Acceleration at time 't' is $a = \frac{dv}{dt} = 24t - 6$

If acceleration is ''0'then $24t - 6 = 0 \Rightarrow t = \frac{1}{4}$

The acceleration of the particle is zero at $t = \frac{1}{4}$.

5. A stone is dropped in to a quiet lake and ripples move in circles at the speed of 5cm/sec. at the instant when the radius of circular ripple in 8 cm, how fast in the enclosed area increases?

Sol: For the circle, let radius = r, area = A

Given $\frac{dr}{dt} = 5, r = 8$

Area A = πr^2 Differentiating w.r.t. 't'we get

$\frac{dA}{dt} = 2\pi e\left(\frac{dr}{dt}\right) = 2\pi(8)(5) = 80\pi$ cm^2/sec.

6. The volume of a cube is increasing at a rate of 9 cubic centimeters per second. How fast is the surface area increasing when the length of the edge is 10 centimeters.

Sol: For the cube, let length of the edge = x

Volume = V

Surface area = S

Given $\frac{dV}{dt}$ = 9 cm³/sec and $x = 10$cm

Volume of the cube $V = x^3$

Differentiating w.r.t. 't', we get $\frac{dV}{dt} = 3x^2\frac{dx}{dt}$

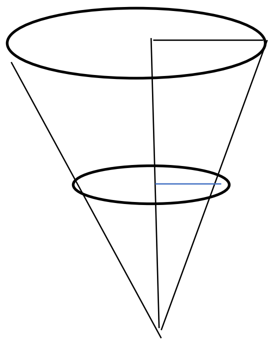
$\Rightarrow 9 = 3x^2\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{9}{3x^2} = \frac{3}{x^2}$

Surface area $S = 6x^2$ differentiating w.r.t. 't' we get

$\frac{dS}{dt} = 12 \times \frac{dx}{dt} = 12x\left(\frac{3}{x^2}\right) = \frac{36}{x} = \frac{36}{10} = 3.6$ cm²/sec.

7. A container in the shape of an inverted cone has height 12cm and radius 6cm at the top. If it is filled with water at the rate of 12cm³/sec, what is the rate of change in the height of water levels when the tank is filled 8cm?

Sol: let OC be the height of water level at 't' sec



Let OC = h, CD = r and volume = V

Given that AB = 6cm, OA = 12 cm, $\frac{dV}{dt}$ = 12cm²/sec

We have to find the rate of rise of the water level

$\left(\frac{dh}{dt}\right)_{h=8}$, when h = 8 cm

The triangle OAB and OCD are similar traingles

$\therefore \frac{CD}{AB} = \frac{OC}{OA} \Rightarrow \frac{r}{6} = \frac{h}{12} \Rightarrow r = \frac{h}{2}$ ----- (2)

Volume of the cone V is given by $V = \frac{\pi r^2 h}{3}$ ------ (2)

From (1), we have $V = \frac{\pi}{3}\left(\frac{h}{2}\right)^3 \times h = \frac{\pi h^3}{12}$ ------ (3)

Diff (3) w.r.t. 't' we get $\frac{dV}{dt} = \frac{\pi h^3}{12}.\frac{dh}{dt}$

$\Rightarrow \frac{dh}{dt} = \frac{4}{\pi h^2}.\frac{dV}{dt} = \frac{1}{\pi}\left(\frac{4}{8^2}\right)(12) = \frac{3}{4\pi}$ cm/sec.

Hence, the rate of change of water level is $\frac{3}{4\pi}$ cm/sec.

8. A point P is moving on the curve $y = 2x^2$. The x-coordinate of P is increasing at the rate of 4 units per second. Find the rate at which the y-coordinate increasing when the point is at (2, 8)

Sol: Let P = (x, y), given $\frac{dx}{dy} = 4$

The equation of the curve is $y = 2x^2$, differentiating w.r.t. 't'

We have $\frac{dy}{dx} = 4x.\frac{dx}{dt}$

∴ A $x = 2, \frac{dy}{dt} = 4.2.4 = 32$

The rate od increase of the y-coordinate is 32 units/sec.

9. The displacement S of a particle travelling in a straight line in 't' seconds in given $S = 45t + 11t^2 - t^3$. Find the time when the particle comes to rest.
10. The radius of an air bubble is increasing at the rate of ½ cm/sec. at what rate is the volume of the bubble increasing when the radius is 1 cm?
11. The volume of a cube is increasing at a rate of 8 cc per second. How fast is the surface area increasing when length of edge is 12cm?
12. A container is in the shape of an inverted cone ahs height 8m and radius 6m at the top. If it is filled with water at the rate $2m^3$/min, how fast is height of water changing when level in 4m?

10.4 MEAN VALUE OF THEOREMS (2Marks)

SYNPOSIS POINTS

- Mean value theorem: (i) if $f(x)$ is continuous on [a, b] and (ii) differentiable on (a, b) (iiii) then there exists a number c between a and b such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. This is also known first mean value theorem (or) Lagrange's Mean value theorem.
- Rolle's Theorem: (i) if a function $f(x)$ is continuous on [a, b] (ii) and f is differentiable on (a, b) (iii) $f(a) = f(b)$ then there exists at least one c ∈ (a, b) such that $f'(c) = 0$

(This is special case of Lagrange's mean value Theorem)

VSAQ (2 Marks)

1. Verify Rolle's theorem for the function $y = f(x) = x^2 + 4$ on [-3, 3]

Sol: Given that $f(x) = x^2 + 4 \Rightarrow f'(x) = 2x$

(i) $f(x)$ is continuous on [-3, 3] and (ii) differentiable in (-3, 3)

(iii) $f(-3) = (-3)^2 + 4 = 9 + 4 = 13$

$f(3) = 3^2 + 4 = 9 + 4 = 13$

$\therefore f(-3) = f(3)$

So, from Rolle's theorem $f'(c) = 0 \Rightarrow 2c = 0 \Rightarrow c = 0 \in (-3, 3)$

∴ Rolle's theorem verified.

2. Verify Rolle's theorem for the function $f: [-3, 8] \rightarrow R$ be defined by $f(x) = x^2 - 5x + 6$.

Sol: (i) $f(x)$ is continuous on [-3, 8]

(ii) $f(x)$ is derivable on (-3, 8)

$f(-3) = 9 + 15 + 6 = 30$, $f(8) = 64 - 40 + 6 = 30$

$\therefore f(-3) = f(8)$

Now $f'(x) = 2x - 5 \Rightarrow f'(c) = 2c - 5 = 0 \Rightarrow c = \frac{5}{2} \in (-3, 8)$

Hence conditions (i), (ii), (iii) satisfied

∴ Rolle's theorem verified

3. Verify Rolle's theorem for the function $f(x) = x(x+3)e^{-x/2}$ on [-3, 0].e

Sol: (i) given function $f(x)$ is continuous on [-3, 0]

(ii) $f(x)$ is differentiable in (-3, 0)

$f(x) = x(x+3)e^{-x/2}$

$\Rightarrow f(-3) = (-3)(-3+3)e^{-\frac{3}{2}} = -3(0)e^{-3/2} = 0$

$\Rightarrow f(0) = (0)(0+3)e^{0/2} = 0$

$\therefore f(-3) = f(0)$

$\therefore f(x)$ satisfies all the 3 conditions of Rolle's theorem.

There exists $c \in (-3, 0)$ such that $f'(c) = 0$

Now $f(x) = x(x+3)e^{-x/2} = (x^2+3x)e^{-x/2}$

$$f'(x) = (x^2+3x)e^{-x/2}.\left(\frac{1}{2}\right) + e^{-\frac{x}{2}}(2x+3)$$

$$= e^{-x/2}\left[\frac{-x^2-3x}{2} + 2x + 3\right] = e^{-x/2}\left[\frac{-x^2-3x+4x+6}{2}\right]$$

$$= e^{-x/2}\left[\frac{-x^2+x+6}{2}\right]$$

$$\therefore f'(c) = 0 \Rightarrow e^{-c/2}\left[\frac{-c^2+c+6}{2}\right] = 0 \Rightarrow -c^2+c+6 = 0$$

$$\Rightarrow c^2 - c - 6 = 0 \Rightarrow (c+2)(c-3) = 0 \Rightarrow c = -2 \text{ or } 3$$

$-2 \in (-3,0)$ hence Rolle's theorem verified

4. Let $f(x) = (x-1)(x-2)(x-3)$ then prove that there is more than one ç'in (1, 3) such that $f'(c) = 0$.

Sol: $f(x)$ is polynomial and hence condition on [1, 3]

Differentiable on (1, 3) and $f(1) = f(3) = 0$

$f(x)$ satisfies all the 3 conditions of Rolle's theorem

$f(x) = (x-1)(x-2)(x-3)$

$$f'(x) = (x-1)(x-2)(1) + (x-1)(x-3)(1) + (x-2)(x-3)(1)$$

$$f'(x) = 3x^2 - 12x + 11$$

∴ By Rolle's Theorem $\exists\ c \in (1,3)$ such that $f'(c) = 0$

$\Rightarrow 3c^2 - 12c + 11 = 0$

$\Rightarrow c = \frac{12 \pm \sqrt{144-132}}{2(3)} = \frac{12 \pm \sqrt{8}}{6} = \frac{12 \pm 2\sqrt{2}}{6} = 2 \pm \frac{1}{\sqrt{3}}$

5. Verify the conditions of Lagrange's mean value theorem for the function $x^2 - 1$ on [2, 3].

Sol: Given $f(x) = x^2 - 1 \Rightarrow f'(x) = 2x$ ------ (1)

(i) $f(x)$ is continuous on [2, 3] (ii) differentiable in (2, 3)

∴ from Lagrange's theorem $f'(c) = \frac{f(3)-f(2)}{3-2}$

$\Rightarrow 2c = \frac{3^2-1-2^2-1}{3-2} = \frac{9-4}{1} = 5$

$\Rightarrow 2c = 5 \Rightarrow c = \frac{5}{2} = 2.5$

$c = 2.5 \in (2, 3)$

Hence, Lagrange's theorem is verified.

PRACTICE QUESTIONS

1. Verify Rolle's Theorem for the function $x^2 - 1$ on [-1,1]
2. Verify Rolle's Theorem for the function $log(x^2 + 2) - log3$ on (-1, 1)
3. Verify Rolle's Theorem for the function $sinx - sin2x$ on [0, π]
4. Verify Lagrange's mean value theorem for the function $f(x) = x^2$ o [2, 4].

10.5 MAXIMA AND MINIMA (7 Marks)

SYNPOSIS POINTS

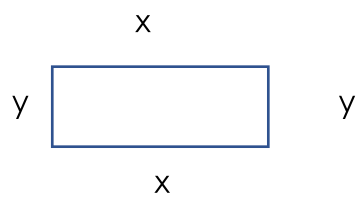
- Critical Point: Critical point of a differentiable function is/are values in HS domain where its derivative is zero or undefined.
- Stationary points: A point x_0, at which $f'(x_0) = 0$
- Point of inflection: A point on a curve at which the sign of the curvature changes.
- Absolute maximum: The maximum value of the function in the given domain (including boundary points) in known as absolute maximum.
- Maxima point of a function: The real valued function $f(x)$ is said to have the maximum value of interval I, if ∋ a point 'a' in I s. t $f(x) \leq f(a) \forall x \in I$.
  The number $f(a)$ is called the maxima or maximum value and the point is called the point of Maxima in I.
- Local Minimum: The minimum value of the function in the given domain (including boundary points) is known as minimum value of the function.
- Minimum point of a function: The real valued function $f(x)$ is said to have minimum value, if ∃ a point $a \in I$ s t $f(x) \geq f(a)$; $\forall\, x \in I$. The number $f(a)$ is called the minima or minimum value of $f(x)$ in I, point is called point of minima of f in I.
- Steps to find maxima or minima
  (i) differentiate $f(x)$ is find $f'(x)$.
  (ii) Find $f''(x)$ is second derivative
  (iii) Find 'a' by using $f'(x) = 0$
  (iv) if $f''(a) < 0$ then 'a' is local maxima, if $f''(a) > 0$ local minima.

LAQ (7Marks)

1. Find the maximum area of the rectangle that can be formed with fixed perimeter 20.

Sol: Let length of rectangle = x

Breadth of rectangle = y



Perimeter = $2(x + y) = 20$

$\Rightarrow x + y = 10$

$\Rightarrow y = 10 - x$ -----(1)

Area of rectangle is $A = xy$

(1) $\Rightarrow A = x(10 - x) = 10x - x^2$ ------ (2)

Differentiating (2) w.r.t. $x \Rightarrow A'(x) = 10 - 2x$ ------ (3)

$$A''(x) = \frac{d}{dx}(10 - 2x) \Rightarrow -2 < 0$$

$\Rightarrow 10 - 2x = 0$

$\Rightarrow x = 5$

Area is maximum at $/x = 5$

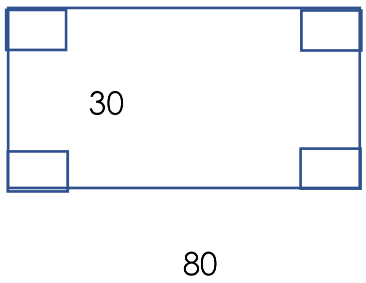
$\therefore y = 10 - x = 10 - 5 = 5 \therefore y = 5.$

$\therefore$ Maximum area of the rectangle is $A = xy = 25$sq. units

2. From a rectangular sheet of dimensions 30 cm × 80 cm, corners, and the sides are then turned up so as to form an open rectangular box. What is the value of x, so that the volume of the box is the greatest?

Sol: Let length of sheet = x

Length = $l = 80 - 2x$



Breadth = $b = 30 - 2x$

Volume V = lbh = $(80 - 2x)(20 - 2x)(x)$

= $2(40 - x)2(15 - x)(x)$

= $4(40 - x)(15 - x)(x)$

= $4(600 - 40x - 15x + x^2)(x)$

= $4(x^3 - 55x^2 + 600x)$

$V(x)$ = $4(x^3 - 55x^2 + 600x)$

Differentiating w.r.t. $'x'$

$V'(x) = 4(3x^2 - 110x + 600)$

Now, $V'(x) = 0 \Rightarrow 4(3x^2 - 110x + 600) = 0$

$\Rightarrow 3x(x - 30) - 20(x - 30) = 0$

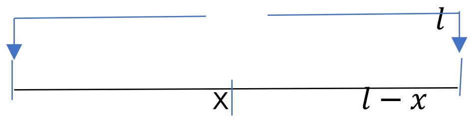
$\Rightarrow x = 20/3$ or 30

At $x = \frac{20}{3}$, $V''\left(\frac{20}{3}\right) = 4\left[6\left(\frac{20}{3}\right) - 110\right] = 4(40 - 110) = 40(-70) = -280 < 0$

$\therefore V(x)$ has maximum value at $x = \frac{20}{3}$cm

3. A wire of length l is cut into two parts which are bent respectively in the form of a square and a circle. What are lengths of pieces of wire so that the sum of areas is least?

Sol: Length of the = l

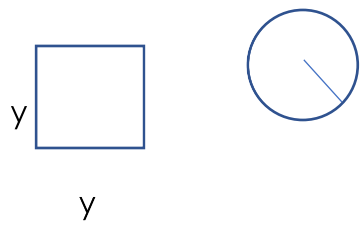


Let x part of l is cut into square of side y

Perimeter $4y = x \Rightarrow y = \frac{x}{4}$ --------- (1)

Remaining part $l - x$ is made into circle of radius $'r'$

Circumference $2\pi r = l - r \Rightarrow r = \frac{l-x}{2\pi}$ ------- (2)



Sum of area square and circle is

$A(x) = y^2 + \pi r^2 = \frac{x^2}{16} + \pi\frac{(l-x)^2}{4\pi^2}$ (from (1) & (2))

$A(x) = \frac{x^2}{16} + \pi\frac{(l-x)^2}{4\pi^2} \Rightarrow A'(x) = \frac{2x}{16} + \frac{2(l-x)}{4\pi} \Rightarrow \frac{x}{8} - \frac{l-x}{2\pi}$ ------ (3)

Now $A'(x) = 0 \Rightarrow \frac{x}{8} - \frac{l-x}{2\pi} = 0$ [$\because$ At max or min $A'(x) = 0$]

$\Rightarrow \frac{x}{8} - \frac{l-x}{2\pi} \Rightarrow \frac{x}{4} - \frac{l-x}{\pi} \Rightarrow x\pi - 4(l - x) \Rightarrow x\pi = 4l - 4x$

$\Rightarrow x\pi + 4x = 4l \Rightarrow x(\pi + 4) = 4l \Rightarrow x = \frac{4l}{\pi+4}$

Also, $l - x = l - \frac{4l}{\pi+4} = \frac{l(\pi+4)-4l}{\pi+4} = \frac{l\pi+4l-4l}{\pi+4} = \frac{\pi l}{\pi+4}$

On differentiating (3) w.r.t. $'x'$ we get $A''(x) = \frac{1}{8} + \frac{1}{2\pi} > 0$

Hence $A(x)$ has a minimum value

$\therefore$ Length of piece that forms square is $\frac{4l}{\pi+4}$ and circle is $\frac{\pi l}{\pi+4}$

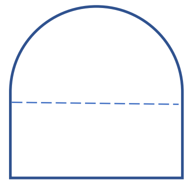
4. A window is in the shape of a rectangle surmounted by a semi-circle. If the perimeter of the window be feet then find the maximum area.

Sol: Length of the rectangle = $2r$

Breadth = x

For the semicircle, radius = r

Semiperimeter = πr



Given total perimeter of the window is 20 feet.

$\Rightarrow 2x + 2r + \pi r = 20 \Rightarrow 2x = 20 - 2r - \pi r$ --------- (1)

Area of the window A = Area of the rectangle + Area of semicircle

$\Rightarrow A(r) = 2r(x) + \frac{\pi r^2}{2} = r(2x) + \frac{\pi r^2}{2}$

$\Rightarrow A(r) = r(20 - 2r - \pi r) + \frac{\pi r^2}{2}$ (from (1))

$= 20r - 2r^2 - \pi r^2 + \frac{\pi r^2}{2} = 20r - r^2\left(2 + \pi - \frac{\pi}{2}\right)$

Diff (2) w.r.t. $'r'$ we get $A'(r) = 20 - 2r\left(\frac{4+\pi}{2}\right)$

$= 20 - r(4 + \pi)$ --------- (3)

Now, $A'(R) = 0 \Rightarrow 20 - r(4 + \pi)$ ($\because$ At max (or) min we have $A'(R) = 0$)

$\Rightarrow r(4 + \pi) = 20 \Rightarrow r = \frac{20}{4+\pi}$ diff (3) w.r.t. $'r'$

We get $A''(r) = 0 - (4 + \pi) < 0$

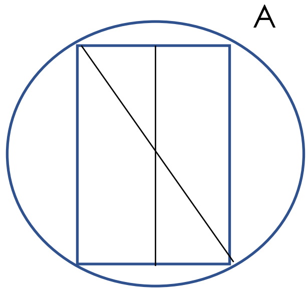
$\therefore A(r)$ has maximum value

$\therefore$ from (2) maximum area is $A = 20\left(\frac{20}{4+\pi}\right) - \frac{400}{(4+\pi)^2}\left(\frac{4+\pi}{2}\right)$

$= \frac{400}{4+\pi} - \frac{200}{4+\pi} = \frac{200}{4+\pi}$ sq. ft

5. If the curved surface right circular cylinder inscribed in a sphere of radius $'r'$ is maximum, show that the height of the cylinder is $\sqrt{2}r$.

Sol: For the sphere, given radius = r (fixed constant)



For the cylinder height = h

Base radius = R

From figure AB = h, BC = 2R, AC = 2r

By pythagorus theorem on ΔABC, $AB^2 + BC^2 = AC^2$

$\Rightarrow h^2 + (2R)^2 = (2r)^2 \Rightarrow h^2 + 4R^2 = 4r^2 \Rightarrow h^2 = 4(r^2 - R^2)$

Curved surface area of the cylinder A = $2\pi rh$

On squaring and taking $A^2 = f(R)$ we get

$f(R) = 4\pi^2R^2h^2 = 4\pi^2R^2\,[4(r^2 - R^2)] = 16\pi^2R^2(r^2 - R^2)$

$= 16\pi^2\,(r^2R^2 - R^4)$

Diff w.r.t. 'R'we get

$f'(R) = 16\pi^2\,(r^2\,2R - 4R^3)$------- (1) ($\because$ r is constant and R is variable)

$\Rightarrow r^2(2R) - 4R^3 = 0 \Rightarrow r^2(2R) - 2R^2(2R) = 0$

$\Rightarrow 2R(r^2 - 2R^2) = 0 \Rightarrow r^2 - 2R^2 = 0 \Rightarrow R^2 = r^2/2$

Now $h^2 = 4(r^2 - R^2) = 4(r^2 - r^2/2) = = 4\left(\frac{2r^2 - r^2}{2}\right) = 2r^2$

$\therefore h^2 = 2r^2 \Rightarrow h = \sqrt{2}r$

Again diff w.r.t. 'R' we get $f''(R) = 16\pi^2[2r^2 - 12R^2]$

At $R^2 = r^2/2$ $\;f''(R) = 16\pi^2[2r^2 - \frac{12r^2}{2}]$

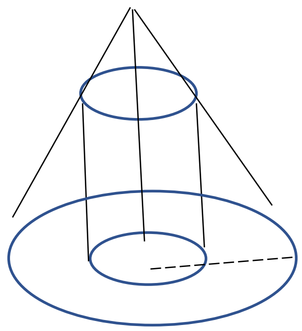
$= 16\pi^2[2r^2 - 6r^2]$

$= 16\pi^2[-4r^2] < 0$

$\therefore$ Surface area is maximum when $h = \sqrt{2}r$

6. Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

Sol: Let centre of the circular base of the cone = 'O'



Height h = AO

Radius of the base r = OC

For cylinder, radius R = OD, height H = FO

$\Rightarrow$ FO = ED = H

$\therefore$ DC = OC – OD = $r - R$

Now ΔAOC and ΔEDC are similar

$\therefore \frac{ED}{AO} = \frac{DC}{OC} \Rightarrow \frac{H}{h} = \frac{r-R}{r}$

$\Rightarrow H = \frac{h(r-R)}{r}$ ---------- (1)

Curved surface area of the cylinder is $S = 2\pi RH$

From (1) $S(R) = \frac{2\pi R(h(r-R))}{r} = \frac{2\pi h(rR-R^2)}{r}$ (r, h are constants)

Now $S'(R) = \frac{2\pi h}{r}(r-2R)$

$\Rightarrow \frac{2\pi h(r-2R)}{r} = 0$ ($\because$ At max. or min value $S'(R) = 0$)

$r - 2R = 0 \Rightarrow R = \frac{r}{2}$

Also $S''(R) = \frac{-4\pi h}{r} < 0, \forall R$

Hence the required radius of the cylinder is $r/2$

7. Find two positive integers whose sum is 15 so that the sum of their squares is minimum.

Sol: Let the two positive numbers be x, y

Given that $x + y = 15 \Rightarrow y = 15 - x$ ------(1)

Let $f(x) = x^2 + y^2 = x^2 + (15-x)^2$

$\therefore f(x) = x^2 + (15-x)^2$ ------- (2)

Diff (2) w.r.t. $'x'$ we get

$f'(x) = 2x + 3(15 - x)(-1) = 2x - 30 + 2x$

$= 4x - 30 = 2(2x - 15)$ ------ (3)

At max. or min. we have $f'(x) = 0$

$2x - 15 = 0 \Rightarrow x = \frac{15}{2}$

Diff (3) w.r.t. $'x'$ we get $f''(x) = 4$ ------- (4)

At $x = \frac{15}{2}$, from (4), $f''\left(\frac{15}{2}\right) = 4 > 0$

$y = 15 - \frac{15}{2} = \frac{15}{2}$ $\therefore f(x)$ is minimum when $x = \frac{15}{2}$ and

$\therefore$ Required number are $(x, y) = \left(\frac{15}{2}, \frac{15}{2}\right)$

PRACTICE QUESTIONS

8. Find two positive integers whose sum is 16 and the sum of whose squares in minimum.
9. Find two positive integers x and y such that $x + y = 60$ and xy^3 is maximum.
10. Find the absolute extremum of $f(x) = x^2$ defined on [-2, 2].
11. Find the points of local extrema for $f(x) = cos4x$ defined (0, $\pi/2$)
12. Determine the intervals in which $f(x) = \frac{2}{x-1} + 18x$, $\forall\ x \in R - \{0\}$ is strictly increasing and decreasing.
13. The profit function $P(x)$ of a company, selling x items per day is given by $P(x) = (150 - x)x - 1600.$ find the number of items that the company should sell to get maximum profit. Also find the maximum profit.
14. What s error, relative error and percentage error in y,
15. Define stationary point of a function.
16. Define Rolle's and Lagrange's Mean value Theorem.