JEE–MAIN EXAMINATION – JANUARY 2025

(HELD ON WEDNESDAY 22nd JANUARY 2025) **TIME : 9:00 AM TO 12:00 NOON**

MATHEMATICS — TEST PAPER WITH SOLUTION

SECTION-A

1. The number of non-empty equivalence relations on the set {1,2,3} is :

(1) 6 (2) 7

(3) 5 (4) 4

Ans. (3)

Sol. Let R be the required relation

A = {(1, 1) (2, 2), (3, 3)}

(i) | R | = 3, when R = A

(ii) | R | = 5, e.g. R = A ∪ {(1, 2), (2, 1)}

Number of R can be [3]

(iii) R = {1, 2, 3} × {1, 2, 3}

Ans. (5)

2. Let f : **R**→**R** be a twice differentiable function such that $f(x + y) = f(x)\, f(y)$ for all x, y ∈ **R**. If $f'(0) = 4a$ and f staisfies $f''(x) - 3a\, f'(x) - f(x) = 0$, a > 0, then the area of the region

R = {(x,y) | $0 \le y \le f(ax)$, $0 \le x \le 2$} is :

(1) $e^2 - 1$ (2) $e^4 + 1$

(3) $e^4 - 1$ (4) $e^2 + 1$

Ans. (1)

Sol. $f(x + y) = f(x).f(y)$

$\Rightarrow f(x) = e^{\lambda x}$ $f'(0) = 4a$

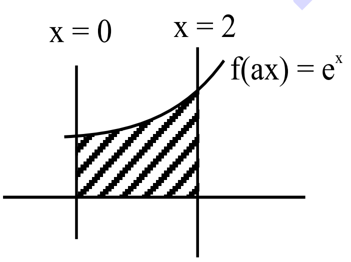
$\Rightarrow f'(x) = \lambda e^{\lambda x}$ $\Rightarrow \lambda = 4a$

So, $f(x) = e^{4ax}$

$f''(x) - 3af'(x) - f(x) = 0$

$\Rightarrow \lambda^2 - 3a\lambda - 1 = 0$

$\Rightarrow 16a^2 - 12a^2 - 1 = 0 \Rightarrow 4a^2 = 1 \Rightarrow \boxed{a = \frac{1}{2}}$



$F(x) = e^{2x}$

$\text{Area} = \int_0^2 e^x dx = \boxed{e^2 - 1}$

3. Let the triangle PQR be the image of the triangle with vertices (1,3), (3,1) and (2, 4) in the line x + 2y = 2. If the centroid of ΔPQR is the point (α, β), then 15(α – β) is equal to :

(1) 24 (2) 19

(3) 21 (4) 22

Ans. (4)

Sol. Let 'G' be the centroid of Δ formed by (1, 3) (3, 1) & (2, 4)

$G \cong \left(2, \frac{8}{3}\right)$

Image of G w.r.t. x + 2y – 2 = 0

$\frac{\alpha - 2}{1} = \frac{\beta - \frac{8}{3}}{2} = -2\frac{\left(2 + \frac{16}{3} - 2\right)}{1+4}$

$= \frac{-2}{5}\left(\frac{16}{3}\right)$

$\Rightarrow \alpha = \frac{-32}{15} + 2 = \frac{-2}{15}, \ \beta = \frac{-32 \times 2}{15} + \frac{8}{3} = \frac{-24}{15}$

$15(\alpha - \beta) = -2 + 24 = 22$

4. Let z_1, z_2 and z_3 be three complex numbers on the circle $|z| = 1$ with $\arg(z_1) = \frac{-\pi}{4}$, $\arg(z_2) = 0$ and $\arg(z_3) = \frac{\pi}{4}$.

If $\left|z_1\bar{z}_2 + z_2\bar{z}_3 + z_3\bar{z}_1\right|^2 = \alpha + \beta\sqrt{2}$, α, β ∈ Z, then the value of $\alpha^2 + \beta^2$ is :

(1) 24 (2) 41

(3) 31 (4) 29

Ans. (4)

Sol. $Z_1 = e^{-i\pi/4}$, $Z_2 = 1$, $Z_3 = e^{i\pi/4}$

$\left|z_1\bar{z}_2 + z_2\bar{z}_3 + z_3\bar{z}_1\right|^2 = \left|e^{-i\frac{\pi}{4}} \times 1 + 1 \times e^{-i\frac{\pi}{4}} + e^{i\frac{\pi}{4}} \times e^{i\frac{\pi}{4}}\right|^2$

$\left|e^{-i\frac{\pi}{4}} + e^{-i\frac{\pi}{4}} + e^{i\frac{\pi}{4}}\right|^2$

$= \left|2e^{-i\frac{\pi}{4}} + i\right|^2 = \left|\sqrt{2} - \sqrt{2}i + i\right|^2$

$= \left(\sqrt{2}\right)^2 + \left(1 - \sqrt{2}\right)^2 = 2 + 1 + 2 - 2\sqrt{2} = 5 - 2\sqrt{2}$

$\alpha = 5, \beta = -2$

$\Rightarrow \alpha^2 + \beta^2 = 29$

5. Using the principal values of the inverse trigonometric functions the sum of the maximum and the minimum values of $16((\sec^{-1}x)^2 + (\text{cosec}^{-1}x)^2)$ is :

(1) $24\pi^2$ (2) $18\pi^2$

(3) $31\pi^2$ (4) $22\pi^2$

Ans. **(4)**

Sol. $16(\sec^{-1}x)^2 + (\text{cosec}^{-1}x)^2$

$\text{Sec}^{-1}x = a \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

$\text{cosec}^{-1}x = \frac{\pi}{2} - a$

$= 16\left[a^2 + \left(\frac{\pi}{2} - a\right)^2\right] = 16\left[2a^2 - \pi a + \frac{\pi^2}{4}\right]$

$\max]_{a=\pi} = 16[2\pi^2 - \pi^2 + \pi\frac{2}{4}] = 20\pi^2$

$\min]_{a=\frac{\pi}{4}} = 16\left[\frac{2\times\pi^2}{16} - \frac{\pi^2}{4} + \frac{\pi^2}{4}\right] = 2\pi^2$

Sum $= 22\pi^2$

6. A coin is tossed three times. Let X denote the number of times a tail follows a head. If μ and σ^2 denote the mean and variance of X, then the value of $64(\mu + \sigma^2)$ is :

(1) 51 (2) 48

(3) 32 (4) 64

Ans. **(2)**

Sol. HHH $\rightarrow$ 0

HHT $\rightarrow$ 0

HTH $\rightarrow$ 1

HTT $\rightarrow$ 0

THH $\rightarrow$ 1

THT $\rightarrow$ 1

TTH $\rightarrow$ 1

TTT $\rightarrow$ 0

Probability distribution

x_i	0	1
$P(x_i)$	$1/2$	$1/2$

$\mu = \sum x_i p_i = \frac{1}{2}$

$\sigma^2 = \sum x_i^2 p_i - \mu^2$

$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

$64(\mu + \sigma^2) = 64\left(\frac{1}{2} + \frac{1}{4}\right) = 48$

7. Let $a_1, a_2, a_3.....$ be a G.P. of increasing positive terms. If $a_1a_5 = 28$ and $a_2 + a_4 = 29$, the a_6 is equal to

(1) 628 (2) 526

(3) 784 (4) 812

Ans. **(3)**

Sol. $a_1.a_5 = 28 \Rightarrow a.ar^4 = 28 \Rightarrow a^2r^4 = 28$ ...(1)

$a_2 + a_4 = 29 \Rightarrow ar + ar^3 = 29$

$\Rightarrow ar(1 + r^2) = 29$

$\Rightarrow a^2r^2(1 + r^2)^2 = (29)^2$ ...(2)

By Eq. (1) & (2)

$\frac{r^2}{(1+r^2)^2} = \frac{28}{29\times29}$

$\Rightarrow \frac{r}{1+r^2} = \frac{\sqrt{28}}{29} \Rightarrow r = \sqrt{28}$

$\because a^2r^4 = 28 \Rightarrow a^2 \times (28)^2 = 28$

$\Rightarrow a = \frac{1}{\sqrt{28}}$

$\therefore a_6 = ar^5 = \frac{1}{\sqrt{28}}\times(28)^2\sqrt{28} = 784$

8. Let $L_1 : \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and

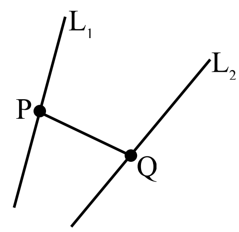
$L_2 : \frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$ be two lines. Then which of the following points lies on the line of the shortest distance between L_1 and L_2 ?

(1) $\left(-\frac{5}{3}, -7, 1\right)$ (2) $\left(2, 3, \frac{1}{3}\right)$

(3) $\left(\frac{8}{3}, -1, \frac{1}{3}\right)$ (4) $\left(\frac{14}{3}, -3, \frac{22}{3}\right)$

Ans. **(4)**

Sol.



$P(2\lambda + 1, 3\lambda + 2, 4\lambda + 3)$ on L_1

$Q(3\mu + 2, 4\mu + 4, 5\mu + 5)$ on L_2

Dr's of PQ $= 3\mu - 2\lambda + 1, 4\mu - 3\lambda + 2, 5\mu - 4\lambda + 2$

$PQ \perp L_1$

$\Rightarrow$ (3μ – 2λ + 1)2 + (4μ – 3λ + 2)3 + (5μ – 4λ + 2)4 = 0

$38\mu - 29\lambda + 16 = 0 \quad \ldots(1)$

$PQ \perp L_2$

$\Rightarrow$ (3μ – 2λ + 1)3 + (4μ – 3λ + 2)4 + (5μ – 4λ + 2)5 = 0

$50\mu - 38\lambda + 21 = 0 \quad \ldots(2)$

By (1) & (2)

$$\lambda = \frac{1}{3};\ \mu = \frac{-1}{6}$$

$$\therefore\ P\left(\frac{5}{3},3,\frac{13}{3}\right)\ \&\ Q\left(\frac{3}{2},\frac{10}{3},\frac{25}{6}\right)$$

Line PQ

$$\frac{x-\frac{5}{3}}{\frac{1}{6}} \qquad \frac{y-3}{\frac{-1}{3}} \qquad \frac{z-\frac{13}{3}}{\frac{1}{6}}$$

$$\frac{x-\frac{5}{3}}{1} = \frac{y-3}{-2} = \frac{z-\frac{13}{3}}{1}$$

$$\text{Point}\left(\frac{14}{3},-3,\frac{22}{3}\right)$$

lies on the line PQ

9. The product of all solutions of the equation $e^{5(\log_e x)^2+3} = x^8$, $x > 0$, is :

(1) $e^{8/5}$ (2) $e^{6/5}$

(3) e^2 (4) e

Ans. (1)

Sol. $e^{5(\ell nx)^2+3} = x^8$

$$\Rightarrow \ell n\, e^{5(\ell nx)^2+3} = \ell n\, x^8$$

$$\Rightarrow 5(\ell nx)^2 + 3 = 8\ell nx$$

$$(\ell nx = t)$$

$$\Rightarrow 5t^2 - 8t + 3 = 0$$

$$t_1 + t_2 = \frac{8}{5}$$

$$\ell n x_1 x_2 = \frac{8}{5}$$

$$x_1 x_2 = e^{8/5}$$

10. If $\sum_{r=1}^{n} T_r = \frac{(2n-1)(2n+1)(2n+3)(2n+5)}{64}$, then $\lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{1}{T_r}\right)$ is equal to :

(1) 1 (2) 0

(3) $\frac{2}{3}$ (4) $\frac{1}{3}$

Ans. (3)

Sol. $T_n = S_n - S_{n-1}$

$$\Rightarrow T_n = \frac{1}{8}(2n-1)(2n+1)(2n+3)$$

$$\Rightarrow \frac{1}{T_n} = \frac{8}{(2n-1)(2n+1)(2n+3)}$$

$$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{T_r} = \lim_{n\to\infty} 8\sum_{r=1}^{n}\frac{1}{(2n-1)(2n+1)(2n+3)}$$

$$= \lim_{n\to\infty}\frac{8}{4}\sum\left(\frac{1}{(2n-1)(2n+1)} - \frac{1}{(2n+1)(2n+3)}\right)$$

$$= \lim_{n\to\infty} 2\left[\left(\frac{1}{1.3}-\frac{1}{3.5}\right)+\left(\frac{1}{3.5}-\frac{1}{5.7}\right)+\ldots\right]$$

$$= \frac{2}{3}$$

11. From all the English alphabets, five letters are chosen and are arranged in alphabetical order. The total number of ways, in which the middle letter is 'M', is :

(1) 14950 (2) 6084

(3) 4356 (4) 5148

Ans. (4)

Sol. $\underbrace{AB\_\_\_}_{12}\ M\ \underbrace{N\ldots\ldots\ldots Z}_{13}$

$$= \underbrace{{}^{12}C_2}_{\text{Selection of two letters before M}} \times \underbrace{{}^{13}C_2}_{\text{Selection of two letters after M}} = 5148$$

12. Let x = x(y) be the solution of the differential equation $y^2dx + \left(x - \frac{1}{y}\right)dy = 0$. If x(1) = 1, then $x\left(\frac{1}{2}\right)$ is :

(1) $\frac{1}{2}+e$ (2) $\frac{3}{2}+e$

(3) 3 – e (4) 3 + e

Ans. **(3)**

Sol. $\frac{dx}{dy}+\left(\frac{1}{y^2}\right)x=\frac{1}{y^3}$

I.F. $= e^{\int \frac{1}{y^2}dy} = e^{-\frac{1}{y}}$

$\Rightarrow x.e^{-\frac{1}{y}} = \int\left(e^{-\frac{1}{t}}\right).\frac{1}{y^3}dy$

Put $-\frac{1}{y}=t$

$+\frac{1}{y^2}dy=dt$

$x.e^{-\frac{1}{y}}=-\int t.e^t dt$

$x.e^{-\frac{1}{y}}=-te^t+e^t+C$

$x.e^{-\frac{1}{y}}=\frac{+1}{y}e^{-\frac{1}{y}}+e^{-\frac{1}{y}}+C$

$x = 1, y = 1$

$\frac{1}{e}=\frac{1}{e}+\frac{1}{e}+C$

$\Rightarrow C=-\frac{1}{e}$

Put $y=\frac{1}{2}$

$\frac{x}{e^2}=\frac{2}{e^2}+\frac{1}{e^2}-\frac{1}{e}$

$x = 3 - e$

13. Let the parabola $y = x^2 + px - 3$, meet the coordinate axes at the points P, Q and R. If the circle C with centre at $(-1, -1)$ passes through the points P, Q and R, then the area of ΔPQR is :

(1) 4 (2) 6

(3) 7 (4) 5

Ans. **(2)**

Sol. $y = x^2 + px - 3$

Let $P(\alpha, 0)$, $Q(\beta, 0)$, $R(0, -3)$

Circle with centre $(-1, -1)$ is $(x + 1)^2 + (y + 1)^2 = r^2$

Passes through $(0, -3)$

$1^2 + (-2)^2 = r^2$]

$r^2 = 5$

$(x + 1)^2 + (y + 1)^2 = 5$

Put $y = 0$

$(x + 1)^2 = 5 - 1$

$(x + 1)^2 = 4$

$x + 1 = \pm 2$

$x = 1$ or $x = -3$

$\therefore$ P(1, 0) and Q(–3,0)

Area of $\Delta PQR = \frac{1}{2}\begin{vmatrix}1 & 0 & 1\\ -3 & 0 & 1\\ 0 & -3 & 1\end{vmatrix} = 6$

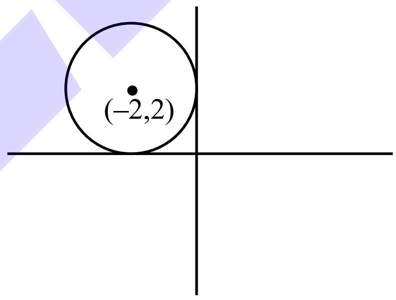
14. A circle C of radius 2 lies in the second quadrant and touches both the coordinate axes. Let r be the radius of a circle that has centre at the point (2, 5) and intersects the circle C at exactly two points. If the set of all possible values of r is the interval (α, β), then $3\beta - 2\alpha$ is equal to :

(1) 15 (2) 14

(3) 12 (4) 10

Ans. **(1)**

Sol.



$S_1 : (x + 2)^2 + (y - 2)^2 = 2^2$

$S_2 : (x - 2)^2 + (y - 5)^2 = r^2$

Both circle intersect at two points

$\therefore |r_1 - r_2| < c_1c_2 < r_1 + r_2$

$|r - 2| < 5 < 2 + r$

$\Rightarrow 3 < r < 7$

$r \in (3, 7)$

$\alpha = 3, \beta = 7$

$3\beta - 2\alpha = 15$

15. Let for $f(x) = 7\tan^8x + 7\tan^6x - 3\tan^4x - 3\tan^2x$, $I_1 = \int_0^{\pi/4} f(x)dx$ and $I_2 = \int_0^{\pi/4} x f(x)dx$. Then $7I_1 + 12I_2$ is equal to :

(1) 2π (2) π

(3) 1 (4) 2

Ans. **(3)**

Sol. $f(x) = (7\tan^6 x - 3\tan^2 x)(\sec^2 x)$

$$I_1 = \int_0^{\pi/4} (7\tan^6 x - 3\tan^2 x)(\sec^2 x)dx$$

Put tanx = t

$$I_1 = \int_0^1 (7t^6 - 3t^2)dt = \left[t^7 - t^3\right]_0^1 = 0$$

$$I_2 = \int_0^{\pi/4} \underset{I}{x}\underbrace{(7\tan^6 x - 3\tan^2 x)(\sec^2 x)}_{II}dx$$

$$= \left[x\left(\tan^7 x - \tan^3 x\right)\right]_0^{\pi/4} - \int_0^{\pi/4} (\tan^7 x - \tan^3 x)dx$$

$$= 0 - \int_0^{\pi/4} \tan^3 x\left(\tan^2 x - 1\right)\left(1 + \tan^2 x\right)dx$$

Put tanx = t

$$= -\int_0^1 \left(t^5 - t^3\right)dt = -\left[\frac{t^6}{6} - \frac{t^4}{4}\right] = \frac{1}{12}$$

$7I_1 + 12I_2 = 1$

16. Let $f(x)$ be a real differentiable function such that $f(0) = 1$ and $f(x + y) = f(x)f'(y) + f'(x) f(y)$ for all $x, y \in \mathbf{R}$. Then $\sum_{n=1}^{100} \log_e f(n)$ is equal to :

(1) 2384 (2) 2525

(3) 5220 (4) 2406

Ans. **(2)**

Sol. $f(x + y) = f(x) f'(y) + f'(x) f(x)$

Put = x = y = 0

$f(0) = f(0)f'(0) + f'(0)f(0)$

$f'(0) = \frac{1}{2}$

Put y = 0

$f(x) = f(x) f'(0) + f'(x)f(0)$

$$f(x) = \frac{1}{2}f(x) + f'(x)$$

$$f'(x) = \frac{f(x)}{2}$$

$$\frac{dy}{dx} = \frac{y}{2} \Rightarrow \int\frac{dy}{y} = \int\frac{dx}{2}$$

$$\Rightarrow \ell ny = \frac{x}{2} + c$$

$\because f(0) = 1 \Rightarrow C = 0$

$$\ell ny = \frac{\pi}{2} \Rightarrow f(x) = e^{x/2}$$

$$\ell n\, f(n) = \frac{n}{2}$$

$$\sum_{n=1}^{100} \ell f(n) = \frac{1}{2}\sum_{n=1}^{100} n = \frac{5050}{2}$$

$= 2525$

17. Let A = {1,2,3,.......,10} and

$$B = \left\{\frac{m}{n} : m, n \in A, m < n \text{ and gcd } (m, n) = 1\right\}.$$

Then n(B) is equal to :

(1) 31 (2) 36

(3) 37 (4) 29

Ans. **(1)**

Sol. A = {1, 2, ….10}

$B\ \{\frac{m}{n} = m, n \in A, m < n, \gcd(m, n) = 1\}$

n(B)

n = 2 $\left\{\frac{1}{2}\right\}$

n = 3 $\left\{\frac{1}{3}, \frac{2}{3}\right\}$

n = 4 $\left\{\frac{1}{4}, \frac{3}{4}\right\}$

n = 5 $\left\{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\right\}$

n = 6 $\left\{\frac{1}{6}, \frac{5}{6}\right\}$

n = 7 $\left\{\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}\right\}$

n = 8 $\left\{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\right\}$

n = 9 $\left\{\frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{5}{9}, \frac{7}{9}, \frac{8}{9}\right\}$

n = 10 $\left\{\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}\right\}$

n(B) = 31

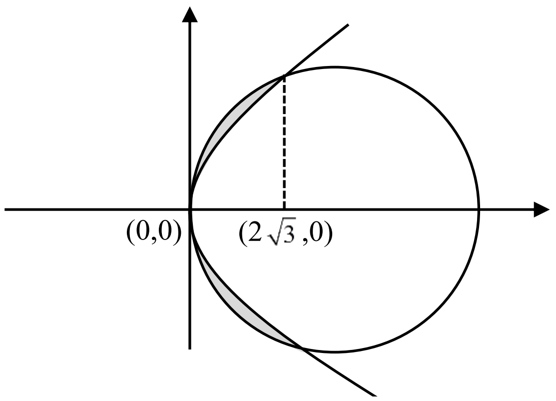
18. The area of the region, inside the circle $\left(x-2\sqrt{3}\right)^2+y^2=12$ and outside the parabola $y^2=2\sqrt{3}x$ is

(1) $6\pi-8$ (2) $3\pi-8$

(3) $6\pi-16$ (4) $3\pi+8$

Ans. (3)

Sol.



$y^2=2\sqrt{3}x$

$\left(x-2\sqrt{3}\right)^2+y^2=\left(2\sqrt{3}\right)^2$

$$A=\frac{\pi r^2}{2}-2\int_0^{2\sqrt{3}}\sqrt{2\sqrt{3}x}\,dx$$

$$\frac{\pi(12)}{2}-2\sqrt{2\sqrt{3}}\frac{\left(x^{3/2}\right)_0^{2/3}}{3/2}$$

$=6\pi-16$

19. Two balls are selected at random one by one without replacement from a bag containing 4 white and 6 black balls. If the probability that the first selected ball is black, given that the second selected ball is also black, is $\frac{m}{n}$, where gcd(m, n) = 1, then m + n is equal to :

(1) 14 (2) 4

(3) 11 (4) 13

Ans. (1)

Sol. $$P=\frac{\frac{6}{10}\times\frac{5}{9}}{\frac{4}{10}\times\frac{6}{9}+\frac{6}{10}\times\frac{5}{9}}=\frac{5}{9}$$

$m=5, n=9$

$m+n=14$

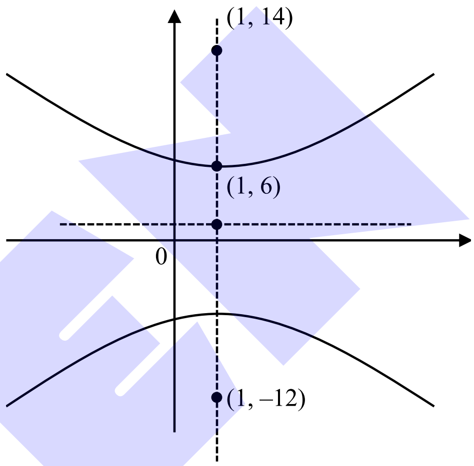
20. Let the foci of a hyperbola be (1, 14) and (1, –12). If it passes through the point (1, 6), then the length of its latus-rectum is :

(1) $\frac{25}{6}$ (2) $\frac{24}{5}$

(3) $\frac{288}{5}$ (4) $\frac{144}{5}$

Ans. (3)

Sol.



$be=13, b=5$

$a^2=b^2(e^2-1)$

$=b^2e^2-b^2$

$=169-25=144$

$$\ell(LR)=\frac{2a^2}{b}=\frac{2\times144}{5}=\frac{288}{5}$$

SECTION-B

21. Let the function,

$$f(x)=\begin{cases}-3ax^2-2, & x<1\\ a^2+bx, & x\geq1\end{cases}$$

Be differentiable for all $x\in\mathbf{R}$, where $a>1$, $b\in\mathbf{R}$. If the area of the region enclosed by $y=f(x)$ and the line $y=-20$ is $\alpha+\beta\sqrt{3}$, $\alpha,\beta,\in Z$, then the value of $\alpha+\beta$ is ____.

Ans. (34)

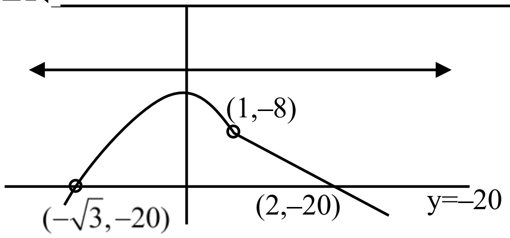
Sol. f(x) is continuous and differentiable

at x = 1; LHL = RHL, LHD = RHD

$-3a-2=a^2+b, -6a=b$

$a=2, 1;\ b=-12$

$$f(x)=\begin{cases}-6x^2-2 & ; \ x<1\\ 4-12x & ; \ x\geq1\end{cases}$$



$$\text{Area} = \int_{-\sqrt{3}}^{1}\left(-6x^2-2+20\right)dx+\int_{1}^{2}(4-12x+20)\,dx$$

$$16+12\sqrt{3}+6=22+12\sqrt{3}$$

22. If $\sum_{r=0}^{5}\frac{{}^{11}C_{2r+1}}{2r+2}=\frac{m}{n}$, gcd(m, n) =1, then m – n is equal to ______.

Ans. **(2035)**

Sol. $\int_0^1(1+x)^{11}dx=\left[C_0x+\frac{C_1x^2}{2}+\frac{C_2x^3}{3}+\ldots\right]_0^1$

$$\frac{2^{12}-1}{12}=C_0+\frac{C_1}{2}+\frac{C_2}{3}+\frac{C_3}{4}+\ldots$$

$$\int_{-1}^{0}(1+x)^{11}dx=\left[C_0x+\frac{C_1x^2}{2}+\frac{C_2x^3}{3}+\ldots\right]_{-1}^{0}$$

$$\frac{1}{12}=C_0-\frac{C_1}{2}+\frac{C_2}{3}-\frac{C_3}{4}+\ldots$$

$$\frac{2^{12}-2}{12}=2\left(\frac{C_1}{2}+\frac{C_3}{4}+\frac{C_5}{6}+\ldots\right)$$

$$\frac{C_1}{2}+\frac{C_3}{4}-\frac{C_5}{6}+\ldots=\frac{2^{11}-1}{12}=\frac{2047}{12}$$

23. Let A be a square matrix of order 3 such that det(A) = –2 and det(3adj(–6adj(3A))) $=2^{m+n}.3^{mn}$, m > n. Then 4m + 2n is equal to _____.

Ans. **(34)**

Sol. $|A|=-2$

det(3adj(–6adj(3A)))

$= 3^3\det(\text{adj}(-\text{adj}(3A)))$

$= 3^3(-6)^6(\det(3A))^4$

$= 3^{21}\times 2^{10}$

m + n = 10

mn = 21

m = 7 ; n = 3

24. Let $L_1:\frac{x-1}{3}=\frac{y-1}{-1}=\frac{z+1}{0}$ and

$L_2:\frac{x-2}{2}=\frac{y}{0}=\frac{z+4}{\alpha}$, $\alpha\in R$, be two lines, which intersect at the point B. If P is the foot of perpendicular from the point A(1, 1, –1) on L_2, then the value of $26\,\alpha(PB)^2$ is ____.

Ans. **(216)**

Sol. Point B

$(3\lambda+1, -\lambda+1, -1)\equiv(2\mu+2, 0, \alpha\mu-4)$

$3\lambda+1=2\mu+2$

$-\lambda+1=0$

$-1=\alpha\mu-4$

$\lambda=1, \mu=1, \alpha=3$

B(4, 0, –1)

Let Point 'P' is $(2\delta+2, 0, 3\delta-4)$

Dr's of AP $<2\delta+1, -1, 3\delta-3>$

$AP\perp L_2\Rightarrow\ \delta=\frac{7}{13}$

$P\left(\frac{40}{13},0,\frac{-31}{13}\right)$

$2\sigma\delta(PB)^2=26\times3\times\left(\frac{144}{169}+\frac{324}{169}\right)$

$= 216$

25. Let $\vec{c}$ be the projection vector of $\vec{b}=\lambda\hat{i}+4\hat{k}$, $\lambda>0$, on the vector $\vec{a}=\hat{i}+2\hat{j}+2\hat{k}$. If $|\vec{a}+\vec{c}|=7$, then the area of the parallelogram formed by the vectors $\vec{b}$ and $\vec{c}$ is ______.

Ans. **(16)**

Sol. $\vec{c}=\left(\frac{\vec{b}\cdot\vec{a}}{|\vec{a}|}\right)\frac{\vec{a}}{|\vec{a}|}$

$=\left(\frac{\lambda+8}{9}\right)\left(\hat{i}+2\hat{j}+2\hat{k}\right)$

$\left|\vec{a}+\vec{c}\right|-=7\ \Rightarrow\lambda=4$

Area of parallelogram

$$=\left|\vec{b}\times\vec{c}\right|=\left|\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\ \frac{4}{3}&\frac{8}{3}&\frac{8}{3}\\ 4&0&4\end{vmatrix}\right|$$

$= 16$

JEE–MAIN EXAMINATION – JANUARY 2025

(HELD ON WEDNESDAY 22nd JANUARY 2025) **TIME : 3 : 00 PM TO 6 : 00 PM**

MATHEMATICS — TEST PAPER WITH SOLUTION

SECTION-A

1. Let α, β, γ and δ be the coefficients of x^7, x^5, x^3 and x respectively in the expansion of $\left(x+\sqrt{x^3-1}\right)^5+\left(x-\sqrt{x^3-1}\right)^5$, x > 1. If u and v satisfy the equations

$\alpha u + \beta v = 18$,

$\gamma u + \delta v = 20$,

then u + v equals :

(1) 5 (2) 4

(3) 3 (4) 8

Ans. (1)

Sol. $\left(x+\sqrt{x^3-1}\right)^5+\left(x-\sqrt{x^3-1}\right)^5$

$= 2\{{}^5C_0.x^5 + {}^5C_2.x^3(x^3-1) + {}^5C_4.x(x^3-1)^2\}$

$= 2\{5x^7 + 10x^6 + x^5 - 10x^4 - 10x^3 + 5x\}$

$\Rightarrow \alpha = 10, \beta = 2, \gamma = -20, \delta = 10$

Now, $10u + 2v = 18$

$-20u + 10v = 20$

$\Rightarrow u = 1, v = 4$

$u + v = 5$

2. In a group of 3 girls and 4 boys, there are two boys B_1 and B_2. The number of ways, in which these girls and boys can stand in a queue such that all the girls stand together, all the boys stand together, but B_1 and B_2 are not adjacent to each other, is :

(1) 144 (2) 72

(3) 96 (4) 120

Ans. (1)

Sol. Total – when B_1 and B_2 are together

$= 2!(3!\ 4!) - 2!\ (3!(3!\ 2!)) = 144$

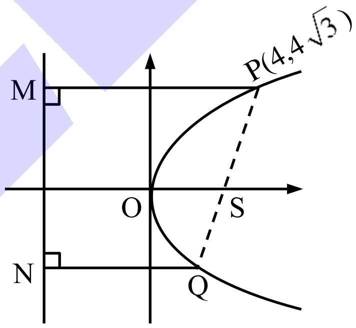
3. Let $P\left(4,4\sqrt{3}\right)$ be a point on the parabola $y^2 = 4ax$ and PQ be a focal chord of the parabola. If M and N are the foot of perpendiculars drawn from P and Q respectively on the directrix of the parabola, then the area of the quadrilateral PQMN is equal to:

(1) $\frac{263\sqrt{3}}{8}$ (2) $17\sqrt{3}$

(3) $\frac{343\sqrt{3}}{8}$ (4) $\frac{34\sqrt{3}}{3}$

Ans. (3)

Sol.



$(4, 4\sqrt{3})$ lies on $y^2 = 4ax$

$\Rightarrow 48 = 4a.4$

$4a = 12$

$\Rightarrow y^2 = 12x$ is equation of parabola

Now, parameter of P is $t_1 = \frac{2}{\sqrt{3}} \Rightarrow$ Parameters of Q is $t_2 = -\frac{\sqrt{3}}{2} \Rightarrow Q\left(\frac{9}{4}, -3\sqrt{3}\right)$

Area of trapezium PQNM

$= \frac{1}{2}MN.(PM + QN)$

$= \frac{1}{2}MN.(PS + QS)$

$= \frac{1}{2}MN.\ PQ$

$= \frac{1}{2}7\sqrt{3}.\frac{49}{4} = (343)\frac{\sqrt{3}}{8} = 3$

4. For a 3 × 3 matrix M, let trace (M) denote the sum of all the diagonal elements of M. Let A be a 3 × 3 matrix such that $|A| = \frac{1}{2}$ and trace (A) = 3. If B = adj(adj(2A)), then the value of |B| + trace (B) equals:

(1) 56 (2) 132
(3) 174 (4) 280

Ans. (4)

Sol. $|A| = \frac{1}{2}$, trace(A) = 3, B = adj(adj(2A)) = $|2A|^{n-2}(2A)$

$n = 3$, $B = |2A|(2A) = 2^3.|A|(2A) = 8A$

$|B| = |8A| = 8^3.|A| = 2^8 = 256$

trace(B) = 8 trace(A) = 24

|B| + trace(B) = 280

5. Suppose that the number of terms in an A.P. is 2k, k∈N. If the sum of all odd terms of the A.P. is 40, the sum of all even terms is 55 and the last term of the A.P. exceeds the first term by 27, then k is equal to

(1) 5 (2) 8
(3) 6 (4) 4

Ans. (1)

Sol. $a_1, a_2, a_3, \ldots\ldots, a_{2k} \xrightarrow{} \text{A.P.}$

$\sum_{r=1}^{k} a_{2r-1} = 40$, $\sum_{r=1}^{k} a_{2r} = 55$, $a_{2k} - a_1 = 27$

$\frac{k}{2}[2a_1 + (k-1)2d] = 40$, $\frac{k}{2}[2a_2 + (k-1)2d] = 55$,

$d = \frac{27}{2k-1}$

$a_1 = \frac{40}{k} - (k-1)d = \frac{55}{k} - kd$

$d = \frac{15}{k} \Rightarrow \frac{27}{2k-1} = \frac{15}{k} \Rightarrow 9k = 10k - 5$

$\therefore k = 5$

6. Let a line pass through two distinct points P(–2, –1, 3) and Q, and be parallel to the vector $3\hat{i} + 2\hat{j} + 2k$. If the distance of the point Q from the point R(1, 3, 3) is 5, then the square of the area of ΔPQR is equal to:

(1) 136 (2) 140
(3) 144 (4) 148

Ans. (1)

Sol. $\overrightarrow{PQ}$ parallel to $3\hat{i} + 2\hat{j} + 2\hat{k}$, R(1, 3, 3)

$\Rightarrow Q(3\lambda - 2, 2\lambda - 1, 2\lambda + 3), \lambda \in R - \{0\}$

$|\overrightarrow{QR}| = 5 = \sqrt{(3\lambda - 3)^2 + (2\lambda - 4)^2 + (2\lambda)^2}$

$\therefore 17\lambda^2 - 34\lambda + 25 = 25 \Rightarrow \lambda = 2 (\because \lambda \neq 0)$

$\therefore$ Q(4, 3, 7), P(–2, –1, 3), R(1, 3, 3)

Area of ΔPQR = [PQR] = $\frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}|$

$$[PQR] = \frac{1}{2}\left\|\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 6 & 4 & 4 \\ 3 & 4 & 0 \end{vmatrix}\right\| = \left\|\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 2 \\ 3 & 4 & 0 \end{vmatrix}\right\|$$

$[PQR] = |-8\hat{i} + 6\hat{j} + 6\hat{k}| = \sqrt{136}$

$\therefore [PQR]^2 = 136$

7. If $\lim_{x \to \infty}\left(\left(\frac{e}{1-e}\right)\left(\frac{1}{e} - \frac{x}{1+x}\right)\right)^x = \alpha$, then the value of $\frac{\log_e \alpha}{1 + \log_e \alpha}$ equals :

(1) e (2) e^{-2}
(3) e^2 (4) e^{-1}

Ans. (1)

Sol. $\alpha = \lim_{x \to \infty}\left(\left(\frac{e}{1-e}\right)\left(\frac{1}{e} - \frac{x}{1+x}\right)\right)^x$ (1^∞ form)

$\therefore \alpha = e^L$

Where $L = \lim_{x \to \infty} x\left(\left(\frac{e}{1-e}\right)\left(\frac{1}{e} - \frac{x}{1+x}\right) - 1\right)$

$\Rightarrow L = \lim_{x \to \infty}\left(\frac{e}{1-e}\right)x\left(\frac{1}{e} - \frac{x}{1+x} - \left(\frac{1-e}{e}\right)\right)$

$\Rightarrow L = \frac{e}{1-e}\lim_{x \to \infty} x\left(1 - \frac{x}{1+x}\right)$

$\Rightarrow L = \frac{e}{1-e}\lim_{x \to \infty}\frac{x}{x+1}$

$\Rightarrow L = \frac{e}{1-e}.1$

$\Rightarrow L = \frac{e}{1-e}$

$\therefore \alpha = e^{\frac{e}{1-e}} \Rightarrow \log\alpha = \frac{e}{1-e}$

$\therefore$ Required value $= \frac{\frac{e}{1-e}}{1 + \frac{e}{1-e}} = e$

8. Let $f(x) = \int_0^{x^2} \frac{t^2 - 8t + 15}{e^t} dt$, $x \in \mathbf{R}$. Then the numbers of local maximum and local minimum points of f, respectively, are :

(1) 2 and 3 (2) 3 and 2

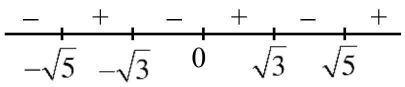
(3) 1 and 3 (4) 2 and 2

Ans. (1)

Sol. $f'(x) = \left(\frac{x^4 - 8x^2 + 15}{e^{x^2}}\right)(2x)$

$= \frac{(x^2-3)(x^2-5)(2x)}{e^{x^2}}$

$= \frac{(x-\sqrt{3})(x+\sqrt{3})(x-\sqrt{5})(x+\sqrt{5})2x}{e^{x^2}}$



Maxima at $x \in \{-\sqrt{3}, \sqrt{3}\}$

Minima at $x \in \{-\sqrt{5}, 0, \sqrt{5}\}$

2 points of maxima and 3 points of minima.

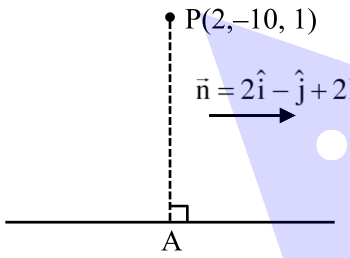
9. The perpendicular distance, of the line $\frac{x-1}{2} = \frac{y+2}{-1} = \frac{z+3}{2}$ from the point P(2, –10,1), is:

(1) 6 (2) $5\sqrt{2}$

(3) $3\sqrt{5}$ (4) $4\sqrt{3}$

Ans. (3)

Sol.



$\frac{x-1}{2} = \frac{y+2}{-1} = \frac{z+3}{2} = \lambda$ (let)

$(2\lambda + 1, -\lambda - 2, 2\lambda - 3)$

$\because \overrightarrow{PA}.\vec{n} = 0$

$\Rightarrow (2\lambda - 1)2 + (-\lambda + 8)(-1) + (2\lambda - 4)2 = 0$

$\Rightarrow 4\lambda - 2 + \lambda - 8 + 4\lambda - 8 = 0$

$\Rightarrow 9\lambda - 18 = 0 \Rightarrow \lambda = 2$

$\therefore$ A(5, –4, 1)

$\therefore AP = \sqrt{3^2 + 6^2 + 0^2} = \sqrt{45} = 3\sqrt{5}$

10. If x = f(y) is the solution of the differential equation

$(1 + y^2) + \left(x - 2e^{\tan^{-1} y}\right)\frac{dy}{dx} = 0$, $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

with f(0) = 1, then $f\left(\frac{1}{\sqrt{3}}\right)$ is equal to :

(1) $e^{\pi/4}$ (2) $e^{\pi/12}$

(3) $e^{\pi/3}$ (4) $e^{\pi/6}$

Ans. (4)

Sol. $\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{2e^{\tan^{-1} y}}{1+y^2}$

I.F. = $e^{\tan^{-1} y}$

$xe^{\tan^{-1} y} = \int \frac{2(e^{\tan^{-1} y})^2 dy}{1+y^2}$

Put $\tan^{-1} y = t$, $\frac{dy}{1+y^2} = dt$

$xe^{\tan^{-1} y} = \int 2e^{2t} dt$

$xe^{\tan^{-1} y} = e^{2\tan^{-1} y} + c$

$x = e^{\tan^{-1} y} + ce^{-\tan^{-1} y}$

$\because$ y = 0, x = 1

1 = 1 + c ⇒ c = 0

$y = \frac{1}{\sqrt{3}}$, $x = e^{\pi/6}$

11. If $\int e^x \left(\frac{x\sin^{-1} x}{\sqrt{1-x^2}} + \frac{\sin^{-1} x}{(1-x^2)^{3/2}} + \frac{x}{1-x^2}\right) dx = g(x) + C$, where C is the constant of integration, then $g\left(\frac{1}{2}\right)$ equals :

(1) $\frac{\pi}{6}\sqrt{\frac{e}{2}}$ (2) $\frac{\pi}{4}\sqrt{\frac{e}{2}}$

(3) $\frac{\pi}{6}\sqrt{\frac{e}{3}}$ (4) $\frac{\pi}{4}\sqrt{\frac{e}{3}}$

Ans. (3)

Sol. $\because \frac{d}{dx}\left(\frac{x\sin^{-1} x}{\sqrt{1-x^2}}\right) = \frac{\sin^{-1} x}{(1-x^2)^{3/2}} + \frac{x}{1-x^2}$

$\Rightarrow \int e^x \left(\frac{x\sin^{-1} x}{\sqrt{1-x^2}} + \frac{\sin^{-1} x}{(1-x^2)^{3/2}} + \frac{x}{1-x^2}\right) dx$

$= e^x . \frac{x\sin^{-1} x}{\sqrt{1-x^2}} + c = g(x) + C$

Note : assuming $g(x) = \frac{xe^x \sin^{-1} x}{\sqrt{1-x^2}}$

$g(1/2) = \frac{e^{1/2}}{2} . \frac{\frac{\pi}{6} \times 2}{\sqrt{3}} = \frac{\pi}{6}\sqrt{\frac{e}{3}}$

Comment : In this question we will not get a unique function g(x), but in order to match the answer we will have to assume $g(x) = \frac{xe^x \sin^{-1} x}{\sqrt{1-x^2}}$.

12. Let α_θ and β_θ be the distinct roots of $2x^2 + (\cos\theta)x - 1 = 0$, $\theta \in (0, 2\pi)$. If m and M are the minimum and the maximum values of $\alpha_\theta^4 + \beta_\theta^4$, then 16(M + m) equals :

(1) 24 (2) 25

(3) 27 (4) 17

Ans. **(2)**

Sol. $(\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2$

$[(\alpha + \beta)^2 - 2\alpha\beta]^2 - 2(\alpha\beta)^2$

$$\left[\frac{\cos^2\theta}{4} + 1\right]^2 - 2.\frac{1}{4}$$

$$\left(\frac{\cos^2\theta}{4} + 1\right)^2 - \frac{1}{2}$$

$$M = \frac{25}{16} - \frac{1}{2} = \frac{17}{16}$$

$$m = \frac{1}{2},\ 16(M + m) = 25$$

13. Let A = {1, 2, 3, 4} and B = {1, 4, 9, 16}. Then the number of many-one functions $f : A \to B$ such that $1 \in f(A)$ is equal to :

(1) 127 (2) 151

(3) 163 (4) 139

Ans. **(2)**

Sol. Total = 4^4

One-one = 4!

Many-one = 256 – 24 = 232

Many-one which $1 \notin f(A)$

= 3.3.3.3 = 81

232 – 81 = 151

14. If the system of linear equations :

$x + y + 2z = 6,$

$2x + 3y + az = a + 1,$

$-x - 3y + bz = 2b,$

where $a, b \in \mathbf{R}$, has infinitely many solutions, then 7a + 3b is equal to :

(1) 9 (2) 12

(3) 16 (4) 22

Ans. **(3)**

Sol. $$\Delta = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 3 & a \\ -1 & -3 & b \end{vmatrix} = 0$$

$\Rightarrow 2a + b - 6 = 0$ ....(1)

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 6 \\ 2 & 3 & a+1 \\ -1 & -3 & 2b \end{vmatrix} = 0$$

$\Rightarrow a + b - 8 = 0$ ....(2)

Solving (1) + (2)

$a = -2,\ b = 10$

$\Rightarrow 7a + 3b = 16$

15. Let $\vec{a}$ and $\vec{b}$ be two unit vectors such that the angle between them is $\frac{\pi}{3}$. If $\lambda\vec{a} + 2\vec{b}$ and $3\vec{a} - \lambda\vec{b}$ are perpendicular to each other, then the number of values of λ in [–1, 3] is :

(1) 3 (2) 2

(3) 1 (4) 0

Ans. **(4)**

Sol. $\hat{a}.\hat{b} = \frac{1}{2}$

Now $\left(\lambda\hat{a} + 2\hat{b}\right).\left(3\hat{a} - \lambda\hat{b}\right) = 0$

$3\lambda\hat{a}.\hat{a} - \lambda^2\hat{a}.\hat{b} + 6\hat{a}.\hat{b} - 2\lambda\hat{b}.\hat{b} = 0$

$$3\lambda - \frac{\lambda^2}{2} + 3 - 2\lambda = 0$$

$\lambda^2 - 2\lambda - 6 = 0$

$\lambda = 1 \pm \sqrt{7}$

$\Rightarrow$ number of values = 0

16. Let E : $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, a > b and H : $\frac{x^2}{A^2}-\frac{y^2}{B^2}=1$. Let the distance between the foci of E and the foci of H be $2\sqrt{3}$. If a – A = 2, and the ratio of the eccentricities of E and H is $\frac{1}{3}$, then the sum of the lengths of their latus rectums is equal to :

(1) 10 (2) 7

(3) 8 (4) 9

Ans. (3)

Sol. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ foci are (ae, 0) and (–ae, 0)

$\frac{x^2}{A^2}+\frac{y^2}{B^2}=1$ foci are (Ae′, 0) and (–Ae′, 0)

$\Rightarrow 2ae = 2\sqrt{3} \Rightarrow ae = \sqrt{3}$

and $2Ae' = 2\sqrt{3} \Rightarrow Ae' = \sqrt{3}$

$\Rightarrow ae = Ae' \Rightarrow \frac{e}{e'}=\frac{A}{a}$

$\Rightarrow \frac{1}{3}=\frac{A}{a} \Rightarrow a = 3A$

Now $a - A = 2 \Rightarrow a - \frac{a}{3} - 2 \Rightarrow a = 3$ and A = 1

$Ae = \sqrt{3} \Rightarrow e = \frac{1}{\sqrt{3}}$ and $e' = \sqrt{3}$

$b^2 = a^2(1 - e^2)$

$b^2 = 6$

and $B^2 = A^2((e')^2 - 1) = (2) \Rightarrow B^2 = 2$

sum of LR $= \frac{2b^2}{a}+\frac{2B^2}{A} = 8$

17. If A and B are two events such that $P(A\cap B) = 0.1$, and $P(A|B)$ and $P(B|A)$ are the roots of the equation $12x^2 - 7x + 1 = 0$, then the value of $\frac{P(\bar{A}\cup\bar{B})}{P(\bar{A}\cap\bar{B})}$ is:

(1) $\frac{5}{3}$ (2) $\frac{4}{3}$

(3) $\frac{9}{4}$ (4) $\frac{7}{4}$

Ans. (3)

Sol. $12x^2 - 7x + 1 = 0$

$x = \frac{1}{3}, \frac{1}{4}$

Let $P\left(\frac{A}{B}\right)=\frac{1}{3}$ & $P\left(\frac{B}{A}\right)=\frac{1}{4}$

$\frac{P(A\cap B)}{P(B)}=\frac{1}{3}$ & $\frac{P(A\cap B)}{P(A)}=\frac{1}{4}$

$\Rightarrow P(B) = 0.3$

& $P(A) = 0.4$

$P(A\cup B) = P(A) + P(B) - P(A\cap B)$

$= 0.3 + 0.4 - 0.1 = 0.6$

Now $\frac{P(\bar{A}\cup\bar{B})}{P(\bar{A}\cap\bar{B})} = \frac{P(\overline{A\cap B})}{P(\overline{A\cup B})}$

$= \frac{1-P(A\cap B)}{1-P(A\cup B)} = \frac{1-0.1}{1-0.6} = \frac{9}{4}$

18. The sum of all values of θ ∈ [0, 2π] satisfying $2\sin^2\theta = \cos 2\theta$ and $2\cos^2\theta = 3\sin\theta$ is

(1) $\frac{\pi}{2}$ (2) 4π

(3) $\frac{5\pi}{6}$ (4) π

Ans. (4)

Sol. $2\sin^2\theta = \cos 2\theta$

$2\sin^2\theta = 1 - 2\sin^2\theta$

$4\sin^2\theta = 1$

$\sin^2\theta = \frac{1}{4}$

$\sin\theta = \pm\frac{1}{2}$

$2\cos^2\theta = 3\sin\theta$

$2 - 2\sin^2\theta + 3\sin\theta - 2 = 0$

$(2\sin\theta - 1)(2\sin\theta - 2) = 0$

$\sin\theta = \frac{1}{2}$

so common equation which satisfy both equations is $\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ (θ ∈ [0, 2π])

Sum = π

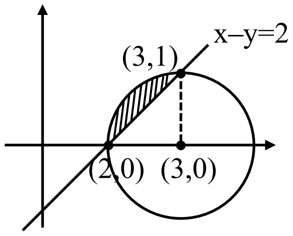
19. Let the curve $z(1+i)+\bar{z}(1-i)=4$, $z \in C$, divide the region $|z-3| \le 1$ into two parts of areas α and β. Then $|\alpha-\beta|$ equals :

(1) $1+\frac{\pi}{2}$ (2) $1+\frac{\pi}{3}$

(3) $1+\frac{\pi}{4}$ (4) $1+\frac{\pi}{6}$

Ans. (1)

Sol.



Let $z = x + iy$

$(x+iy)(1+i)+(x-iy)(1-i)=4$

$x+ix+iy-y+x-ix-iy-y=4$

$2x-2y=4$

$x-y=2$

$|z-3| \le 1$

$(x-3)^2+y^2 \le 1$

Area of shaded region $= \frac{\pi.1^2}{4}-\frac{1}{2}.1.1 = \frac{\pi}{4}-\frac{1}{2}$

Area of unshaded region inside the circle

$= \frac{3}{4}\pi.1^2+\frac{1}{2}.1.1 = \frac{3\pi}{4}+\frac{1}{2}$

$\therefore$ difference of area $= \left(\frac{3\pi}{4}+\frac{1}{2}\right)-\left(\frac{\pi}{4}-\frac{1}{2}\right)$

$= \frac{\pi}{2}+1$

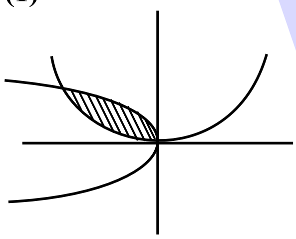
20. The area of the region enclosed by the curves $y = x^2 - 4x + 4$ and $y^2 = 16 - 8x$ is :

(1) $\frac{8}{3}$ (2) $\frac{4}{3}$

(3) 5 (4) 8

Ans. (1)

Sol.



$y=(x-2)^2$, $y^2=8(x-2)$

$y = x^2$, $y^2 = -8x$

$= \frac{16ab}{3} = \frac{16\times\frac{1}{4}\times 2}{3} = \frac{8}{3}$

SECTION-B

21. Let $y = f(x)$ be the solution of the differential equation $\frac{dy}{dx}+\frac{xy}{x^2-1}=\frac{x^6+4x}{\sqrt{1-x^2}}$, $-1 < x < 1$ such that $f(0) = 0$. If $6\int_{-1/2}^{1/2} f(x)dx = 2\pi-\alpha$ then α^2 is equal to __________.

Ans. (27)

Sol. I.F. $e^{-\frac{1}{2}\int\frac{2x}{1-x^2}dx} = e^{-\frac{1}{2}\ell n(1-x^2)} = \sqrt{1-x^2}$

$y\times\sqrt{1-x^2} = \int(x^6+4x)dx = \frac{x^7}{7}+2x^2+c$

Given $y(0) = 0 \Rightarrow c = 0$

$y = \frac{\frac{x^7}{7}+2x^2}{\sqrt{1-x^2}}$

Now, $6\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\frac{x^7}{7}+2x^2}{\sqrt{1-x^2}}dx = 6\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{2x^2}{\sqrt{1-x^2}}dx$

$= 24\int_0^{\frac{1}{2}}\frac{x^2}{\sqrt{1-x^2}}dx$

Put $x = \sin\theta$

$dx = \cos\theta\, d\theta$

$= 24\int_0^{\frac{\pi}{6}}\frac{\sin^2\theta}{\cos\theta}\cos\theta d\theta$

$= 24\int_0^{\frac{\pi}{6}}\left(\frac{1-\cos2\theta}{2}\right)d\theta = 12\left[\theta-\frac{\sin2\theta}{2}\right]_0^{\frac{\pi}{6}}$

$= 12\left(\frac{\pi}{6}-\frac{\sqrt{3}}{4}\right)$

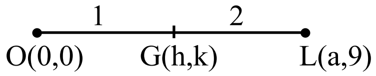
$= 2\pi-3\sqrt{3}$

$\alpha^2 = (3\sqrt{3})^2 = 27$

22. Let A(6, 8), B(10 $\cos\alpha$, $-10\sin\alpha$) and C ($-10\sin\alpha$, $10\cos\alpha$), be the vertices of a triangle. If L(a, 9) and G(h, k) be its orthocenter and centroid respectively, then $(5a - 3h + 6k + 100\sin2\alpha)$ is equal to____________

Ans. (145)

Sol. All the three points A, B, C lie on the circle $x^2 + y^2 = 100$ so circumcentre is (0, 0)



$$\frac{a+0}{3} = h \Rightarrow a = 3h$$

and $\frac{9+0}{3} = k \Rightarrow k = 3$

also centroid $\frac{6+10\cos\alpha - 10\sin\alpha}{3} = h$

$\Rightarrow 10(\cos\alpha - \sin\alpha) = 3h - 6$ ....(i)

and $\frac{8+10\cos\alpha - 10\sin\alpha}{3} = k$

$\Rightarrow 10(\cos\alpha - \sin\alpha) = 3k - 8 = 9 - 8 = 1$....(ii)

on squaring $100(1 - \sin 2\alpha) = 1$

$\Rightarrow 100\sin 2\alpha = 99$

from equ. (i) and (ii) we get $h = \frac{7}{3}$

Now $5a - 3h + 6k + 100\sin 2\alpha$

$= 15h - 3h + 6k + 100\sin 2\alpha$

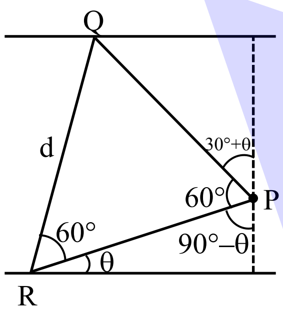
$= 12 \times \frac{7}{3} + 18 + 99$

$= 145$

23. Let the distance between two parallel lines be 5 units and a point P lie between the lines at a unit distance from one of them. An equilateral triangle PQR is formed such that Q lies on one of the parallel lines, while R lies on the other. Then $(QR)^2$ is equal to ________.

Ans. (28)

Sol.



$PR = \text{cosec}\theta$, $PQ = 4\sec(30 + \theta)$

For equilateral

$d = PR = PQ$

$\Rightarrow \cos(\theta + 30°) = 4\sin\theta$

$$\Rightarrow \frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta = 4\sin\theta$$

$$\Rightarrow \tan\theta = \frac{1}{3\sqrt{3}}$$

$QR^2 = d^2 = \text{cosec}^2\theta = 28$

24. If $\sum_{r=1}^{30} \frac{r^2\left({}^{30}C_r\right)^2}{{}^{30}C_{r-1}} = \alpha \times 2^{29}$, then α is equal to ______.

Ans. (465)

Sol. $\sum_{r=1}^{30} \frac{r^2\left({}^{30}C_r\right)^2}{{}^{30}C_{r-1}}$

$$= \sum_{r=1}^{30} r^2\left(\frac{31-r}{r}\right) \cdot \frac{30!}{r!(30-r)!}$$

$$\left(\because \frac{{}^{30}C_r}{{}^{30}C_{r-1}} = \frac{30-r+1}{r} = \frac{31-r}{r}\right)$$

$$= \sum_{r=1}^{30} \frac{(31-r)30!}{(r-1)!(30-r)!}$$

$$= 30\sum_{r=1}^{30} \frac{(31-r)29!}{(r-1)!(30-r)!}$$

$$= 30\sum_{r=1}^{30} (30-r+1)\,{}^{29}C_{30-r}$$

$$= 30\left(\sum_{r=1}^{30}(31-r)\,{}^{29}C_{30-r} + \sum_{r=1}^{30} {}^{29}C_{30-r}\right)$$

$= 30(29 \times 2^{28} + 2^{29}) = 30(29 + 2)2^{28}$

$= 15 \times 31 \times 2^{29}$

$= 465(2^{29})$

$\alpha = 465$

25. Let A = {1, 2, 3}. The number of relations on A, containing (1, 2) and (2, 3), which are reflexive and transitive but not symmetric, is ______.

Ans. (3)

Sol. Transitivity

$(1, 2) \in R, (2, 3) \in R \Rightarrow (1, 3) \in R$

For reflexive $(1, 1), (2, 2)\ (3, 3) \in R$

Now (2, 1), (3, 2), (3, 1)

(3, 1) cannot be taken

(1) (2, 1) taken and (3, 2) not taken

(2) (3, 2) taken and (2, 1) not taken

(3) Both not taken

therefore 3 relations are possible.

JEE–MAIN EXAMINATION – JANUARY 2025

(HELD ON THURSDAY 23rd JANUARY 2025) **TIME : 9:00 AM TO 12:00 NOON**

MATHEMATICS | TEST PAPER WITH SOLUTION

SECTION-A

1. The value of $\int_{e^2}^{e^4} \frac{1}{x}\left(\frac{e^{\left((\log_e x)^2+1\right)^{-1}}}{e^{\left((\log_e x)^2+1\right)^{-1}}+e^{\left((6-\log_e x)^2+1\right)^{-1}}}\right)dx$ is

(1) $\log_e 2$ (2) 2

(3) 1 (4) e^2

Ans. (3)

Sol. Let $\ln x = t \Rightarrow \frac{dx}{x} = dt$

$$I = \int_2^4 \frac{e^{\frac{1}{1+t^2}}}{e^{\frac{1}{1+t^2}} + e^{\frac{1}{1+(6-t)^2}}}dt$$

$$I = \int_2^4 \frac{e^{\frac{1}{1+(6-t)^2}}}{e^{\frac{1}{1+(6-t)^2}} + e^{\frac{1}{1+t^2}}}dt$$

$$2I = \int_2^4 dt = (t)_2^4 = 4-2 = 2$$

$I = 1$

2. Let $I(x) = \int \frac{dx}{(x-11)^{\frac{11}{13}}(x+15)^{\frac{15}{13}}}$.

If $I(37) - I(24) = \frac{1}{4}\left(\frac{1}{b^{\frac{1}{13}}} - \frac{1}{c^{\frac{1}{13}}}\right)$, b, c $\in \mathbb{N}$, then

3(b + c) is equal to

(1) 40 (2) 39

(3) 22 (4) 26

Ans. (2)

Sol. $I(x) = \int \frac{dx}{(x-11)^{\frac{11}{13}}(x+15)^{\frac{15}{13}}}$

Put $\frac{x-11}{x+15} = t \Rightarrow \frac{26}{(x+5)^2}dx = dt$

$$I(x) = \frac{1}{26}\int \frac{dt}{t^{11/13}} = \frac{1}{26}\cdot\frac{t^{2/13}}{2/13}$$

$$I(x) = \frac{1}{4}\left(\frac{x-11}{x+15}\right)^{2/13} + C$$

$$I(37) - I(24) = \frac{1}{4}\left(\frac{26}{52}\right)^{2/13} - \frac{1}{4}\left(\frac{13}{39}\right)^{2/13}$$

$$= \frac{1}{4}\left(\frac{1}{2^{2/13}} - \frac{1}{3^{2/13}}\right)$$

$$= \frac{1}{4}\left(\frac{1}{4^{1/13}} - \frac{1}{9^{1/13}}\right)$$

$\therefore$ b = 4, c = 9

3(b + c) = 39

3. If the function

$$f(x) = \begin{cases} \frac{2}{x}\{\sin(k_1+1)x + \sin(k_2-1)x\} & , x<0 \\ 4 & , x=0 \\ \frac{2}{x}\log_e\left(\frac{2+k_1x}{2+k_2x}\right) & , x>0 \end{cases}$$

is continuous at x = 0, then $k_1^2 + k_2^2$ is equal to

(1) 8 (2) 20

(3) 5 (4) 10

Ans. (4)

Sol. $\lim_{x\to 0^-} \frac{2}{x}\{\sin(k_1+1)x + \sin(k_2-1)x\} = 4$

$\Rightarrow 2(k_1 + 1) + 2(k_2 - 1) = 4$

$\Rightarrow k_1 + k_2 = 2$

$\Rightarrow \lim_{x\to 0^+} \frac{2}{x}\ln\left(\frac{2+k_1x}{2+k_2x}\right) = 4$

$\Rightarrow \lim_{x\to 0^+} \frac{1}{x}\ln\left(1+\frac{(k_1-k_2)x}{2+k_2x}\right) = 2$

$\Rightarrow \frac{k_1-k_2}{2} = 2$

$\Rightarrow k_1 - k_2 = 4$

$\therefore k_1 = 3, k_2 = -1$

$k_1^2 + k_2^2 = 9 + 1 = 10$

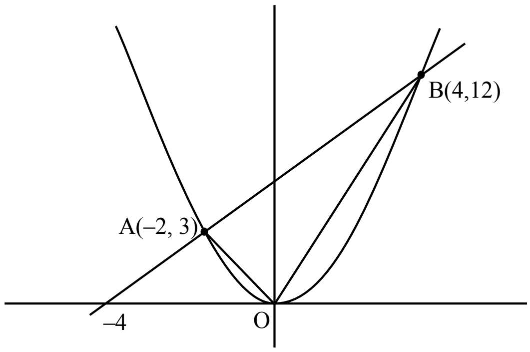
4. If the line $3x - 2y + 12 = 0$ intersects the parabola $4y = 3x^2$ at the points A and B, then at the vertex of the parabola, the line segment AB subtends an angle equal to

(1) $\tan^{-1}\left(\frac{11}{9}\right)$ (2) $\frac{\pi}{2} - \tan^{-1}\left(\frac{3}{2}\right)$

(3) $\tan^{-1}\left(\frac{4}{5}\right)$ (4) $\tan^{-1}\left(\frac{9}{7}\right)$

Ans. (4)

Sol.



$3x - 2y + 12 = 0$

$4y = 3x^2$

$\therefore\ 2(3x + 12) = 3x^2$

$\Rightarrow x^2 - 2x - 8 = 0$

$\Rightarrow x = -2, 4$

$m_{OA} = -3/2,\ m_{OB} = 3$

$$\tan\theta = \left(\frac{\frac{-3}{2} - 3}{1 - \frac{9}{2}}\right) = \frac{9}{7}$$

$\theta = \tan^{-1}\left(\frac{9}{7}\right)$ (angle will be acute)

5. Let a curve $y = f(x)$ pass through the points $(0, 5)$ and $(\log_e 2, k)$. If the curve satisfies the differential equation $2(3 + y)e^{2x}dx - (7 + e^{2x})dy = 0$, then k is equal to

(1) 16 (2) 8

(3) 32 (4) 4

Ans. (2)

Sol. $$\frac{dy}{dx} = \frac{2(3+y).e^{2x}}{7+e^{2x}}$$

$$\frac{dy}{dx} - \frac{2y.e^{2x}}{7+e^{2x}} = \frac{6.e^{2x}}{7+e^{2x}}$$

$$\text{I.F.} = e^{-\int \frac{2e^{2x}}{7+e^{2x}}dx} = \frac{1}{7+e^{2x}}$$

$$\therefore\ y.\frac{1}{7+e^{2x}} = \int \frac{6e^{2x}}{\left(7+3^{2x}\right)^2}dx$$

$$\frac{y}{7+e^{2x}} = \frac{-3}{7+e^{2x}} + C$$

$(0, 5) \Rightarrow \frac{5}{8} = \frac{-3}{8} + C \Rightarrow C = 1$

$\therefore\ y = -3 + 7 + e^{2x}$

$y = e^{2x} + 4$

$\therefore\ k = 8$

6. Let $f(x) = \log_e x$ and $g(x) = \frac{x^4 - 2x^3 + 3x^2 - 2x + 2}{2x^2 - 2x + 1}$. Then the domain of fog is

(1) $\mathbb{R}$ (2) $(0, \infty)$

(3) $[0, \infty)$ (4) $[1, \infty)$

Ans. (1)

Sol. $f(x) = \ln x$

$$g(x) = \frac{x^4 - 2x^3 + 3x^2 - 2x + 2}{2x^2 - 2x + 1}$$

$D_g \in R$

$D_f \in (0, \infty)$

For $D_{fog} \Rightarrow g(x) > 0$

$$\frac{x^4 - 2x^3 + 3x^2 - 2x + 2}{2x^2 - 2x + 1} > 0$$

$\Rightarrow x^4 - 2x^3 + 3x^2 - 2x + 2 > 0$

Clearly $x < 0$ satisfies which are included in option (1) only.

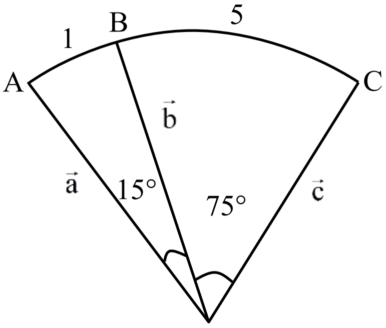
7. Let the arc AC of a circle subtend a right angle at the centre O. If the point B on the arc AC, divides the arc AC such that $\frac{\text{length of arc AB}}{\text{length of arc BC}} = \frac{1}{5}$, and $\overrightarrow{OC} = \alpha\overrightarrow{OA} + \beta\overrightarrow{OB}$, then $\alpha = \sqrt{2}\left(\sqrt{3}-1\right)\beta$ is equal to

(1) $2-\sqrt{3}$ (2) $2\sqrt{3}$

(3) $5\sqrt{3}$ (4) $2+\sqrt{3}$

Ans. **(1)**

Sol.



$\vec{c} = \alpha\vec{a} + \beta\vec{b}$ ....(1)

$\vec{a}.\vec{c} = \alpha\vec{a}.\vec{a} + \beta\vec{b}.\vec{a}$

$0 = \alpha + \beta \cos 15°$ ....(2)

(1) $\Rightarrow \vec{b}.\vec{c} = \alpha\vec{a}.\vec{b} + \beta\vec{b}.\vec{b}$

$\Rightarrow \cos 75° = \alpha \cos 15° + \beta$ ....(3)

(2) & (3) $\Rightarrow \cos 75° = -\beta \cos^2 15° + \beta$

$$\beta = \frac{\cos 75°}{\sin^2 15°} = \frac{1}{\sin 15°} = \frac{2\sqrt{2}}{\sqrt{3}-1}$$

$$(2) \Rightarrow \alpha = \frac{-\cos 15°}{\sin 15°} = \frac{-(\sqrt{3}+1)}{(\sqrt{3}-1)}$$

$$\therefore \vec{c} = \frac{-(\sqrt{3}+1)}{(\sqrt{3}-1)}\vec{a} + \left(\frac{2\sqrt{2}}{\sqrt{3}-1}\right)\vec{b}$$

Now

$$\alpha + \sqrt{2}(\sqrt{3}-1)\beta = \frac{-(\sqrt{3}+1)}{(\sqrt{3}-1)} + \frac{\sqrt{2}(\sqrt{3}-1).2\sqrt{2}}{\sqrt{3}-1}$$

$$= \frac{-(\sqrt{3}+1)^2}{2} + 4$$

$$= \frac{-3-1-2\sqrt{3}+8}{2}$$

$$= 2-\sqrt{3}$$

8. If the first term of an A.P. is 3 and the sum of its first four terms is equal to one-fifth of the sum of the next four terms, then the sum of the first 20 terms is equal to

(1) –1200 (2) –1080

(3) –1020 (4) –120

Ans. **(2)**

Sol. $a = 3$

$S_4 = \frac{1}{5}(S_8 - S_4)$

$\Rightarrow 5S_4 = S_8 - S_4$

$\Rightarrow 6S_4 = S_8$

$\Rightarrow 6.\frac{4}{2}[2\times3+(4-1)\times d]$

$= \frac{8}{2}[2\times3+(8-1)d]$

$\Rightarrow 12(6 + 3d) = 4(6 + 7d)$

$\Rightarrow 18 + 9d = 6 + 7d$

$\Rightarrow d = -6$

$S_{20} = \frac{20}{2}[2\times3+(20-1)(-6)]$

$= 10\,[6 - 114]$

$= -1080$

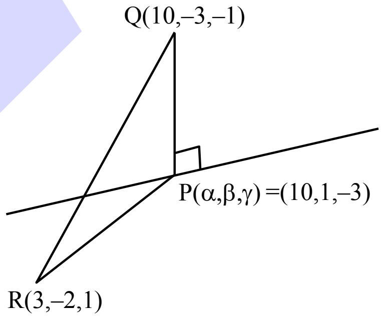
9. Let P be the foot of the perpendicular from the point Q(10, –3, –1) on the line $\frac{x-3}{7} = \frac{y-2}{-1} = \frac{z+1}{-2}$. Then the area of the right angled triangle PQR, where R is the point (3, –2, 1), is

(1) $9\sqrt{15}$ (2) $\sqrt{30}$

(3) $8\sqrt{15}$ (4) $3\sqrt{30}$

Ans. **(4)**

Sol.



$$\frac{x-3}{7} = \frac{y-2}{-1} = \frac{z+1}{-2} = \lambda$$

$\Rightarrow 7\lambda + 3, -\lambda + 2, -2\lambda - 1$

dr's of QP $\Rightarrow$

$7\lambda - 7, -\lambda + 5, -2\lambda$

Now

$(7\lambda - 7) . 7 - (-\lambda + 5) + (2\lambda).2 = 0$

$54\lambda - 54 = 0 \Rightarrow \lambda = 1$

$\therefore$ P = (10, 1, – 3)

$\overrightarrow{PQ} = -4\hat{j} + 2\hat{k}$

$\overrightarrow{PR} = -7\hat{i} - 3\hat{j} + 4\hat{k}$

$$\text{Area} = \left|\frac{1}{2}\begin{vmatrix} i & j & k \\ 0 & -4 & 2 \\ -7 & -3 & 4 \end{vmatrix}\right| = 3\sqrt{30}$$

10. Let $\left|\frac{\bar{z}-i}{2\bar{z}+i}\right| = \frac{1}{3}$, $z \in \mathbb{C}$, be the equation of a circle with center at C. If the area of the triangle, whose vertices are at the points (0, 0), C and (α, 0) is 11 square units, then α^2 equals

(1) 100 (2) 50

(3) $\frac{121}{25}$ (4) $\frac{81}{25}$

Ans. (1)

Sol. $\left|\frac{\bar{z}-i}{2\bar{z}+i}\right| = \frac{1}{3}$

$$\left|\frac{\bar{z}-i}{\bar{z}+\frac{i}{2}}\right| = \frac{2}{3}$$

$$3|x - iy - i| = 2\left|x - iy + \frac{i}{2}\right|$$

$9(x^2 + (y+1)^2) = 4(x^2 + (y - 1/3)^2)$

$9x^2 + 9y^2 + 18y + 9 = 4x^2 + 4y^2 - 4y + 1$

$5x^2 + 5y^2 + 22y + 8 = 0$

$x^2 + y^2 + \frac{22}{5}y + \frac{8}{5} = 0$

centre $\Rightarrow \left(0, -\frac{11}{5}\right)$

$$\left|\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 0 & -11/5 & 1 \\ \alpha & 0 & 1 \end{vmatrix}\right| = 11$$

$$\Rightarrow \left(-\frac{11}{5}\alpha\right)^2 = (11 \times 2)^2$$

$\Rightarrow \alpha^2 = 100$

11. Let R = {(1, 2), (2, 3), (3,3)} be a relation defined on the set {1, 2, 3, 4}. Then the minimum number of elements, needed to be added in R so the R becomes an equivalence relation, is :

(1) 10 (2) 8

(3) 9 (4) 7

Ans. (4)

Sol. A = {1, 2, 3, 4}

For relation to be reflexive

R = {(1,2), (2, 3), (3,3)}

Minimum elements added will be

(1,1), (2,2), (4,4) (2,1) (3,2) (3,2) (3,1) (1,3)

∴ Minimum number of elements = 7

Option : (4)

12. The number of words, which can be formed using all the letters of the word "DAUGHTER", so that all the vowels never come together, is

(1) 34000 (2) 37000

(3) 36000 (4) 35000

Ans. (3)

Sol. DAUGHTER

Total words = 8!

Total words in which vowels are together = 6! × 3!

words in which all vowels are not together

= 8! – 6! × 3!

= 6! [56 – 6]

= 720 × 50

= 36000

Ans.(3)

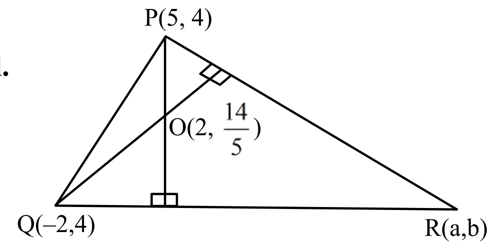
13. Let the area of a ΔPQR with vertices P(5, 4), Q(–2, 4) and R(a, b) be 35 square units. If its orthocenter and centroid are $O\left(2, \frac{14}{5}\right)$ and C(c, d) respectively, then c + 2d is equal to

(1) $\frac{7}{3}$ (2) 3

(3) 2 (4) $\frac{8}{3}$

Ans. (2)

Sol.



Equation of lines QR = 5x + 2y + 2 = 0

Equation of lines PR = 10x –3y – 38 = 0

∴ Point R (2, –6)

$$\text{Centroid} = \left(\frac{5-2+2}{3}, \frac{4+4-6}{3}\right)$$

$$= \left(\frac{5}{3}, \frac{2}{3}\right)$$

$$c + 2d = \frac{5}{3} + \frac{4}{3} = 3$$

14. If $\frac{\pi}{2} \le x \le \frac{3\pi}{4}$, then $\cos^{-1}\left(\frac{12}{13}\cos x + \frac{5}{13}\sin x\right)$ is equal to

(1) $x - \tan^{-1}\frac{4}{3}$ (2) $x - \tan^{-1}\frac{5}{12}$

(3) $x + \tan^{-1}\frac{4}{5}$ (4) $x + \tan^{-1}\frac{5}{12}$

Ans. (2)

Sol. $\frac{\pi}{2} \le x \le \frac{3\pi}{4}$

$\cos^{-1}\left(\frac{12}{13}\cos x + \frac{5}{12}\sin x\right)$

$\cos^{-1}(\cos x \cos\alpha + \sin x \sin\alpha)$

$\cos^{-1}(\cos(x-\alpha))$

$\Rightarrow x - \alpha$ because $x - \alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$\Rightarrow x - \tan^{-1}\frac{5}{12}$.

15. The value of $(\sin 70°)(\cot 10° \cot 70° - 1)$ is

(1) 1 (2) 0

(3) 3/2 (4) 2/3

Ans. (1)

Sol. $\sin 70^{\circ}(\cot 10^{\circ} \cot 70^{\circ} - 1)$

$\Rightarrow \frac{\cos(80°)}{\sin 10} = 1$

16. Marks obtains by all the students of class 12 are presented in a frequency distribution with classes of equal width. Let the median of this grouped data be 14 with median class interval 12-18 and median class frequency 12. If the number of students whose marks are less than 12 is 18, then the total number of students is

(1) 48 (2) 44

(3) 40 (4) 52

Ans. (2)

Sol. medain $= \ell + \left(\frac{\frac{N}{2} - F}{f}\right) \times h$

$= 12 + \left(\frac{\frac{N}{2} - 18}{12}\right) \times 6 = 14$

$\Rightarrow \left(\frac{\frac{N}{2} - 18}{12}\right) \times 6 = 2$

$\frac{N}{2} - 18 = 4 \Rightarrow N = 44$

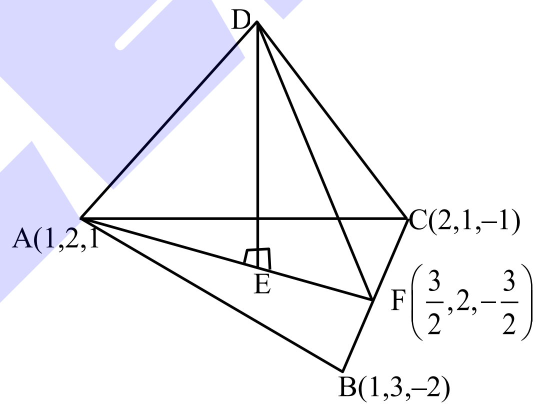
17. Let the position vectors of the vertices A, B and C of a tetrahedron ABCD be $\hat{i} + 2\hat{j} + \hat{k}$, $\hat{i} + 3\hat{j} - 2\hat{k}$ and $2\hat{i} + \hat{j} - \hat{k}$ respectively. The altitude from the vertex D to the opposite face ABC meets the median line segment through A of the triangle ABC at the point E. If the length of AD is $\frac{\sqrt{110}}{3}$ and the volume of the tetrahedron is $\frac{\sqrt{805}}{6\sqrt{2}}$, then the position vector of E is

(1) $\frac{1}{2}(\hat{i} + 4\hat{j} + 7\hat{k})$ (2) $\frac{1}{12}(7\hat{i} + 4\hat{j} + 3\hat{k})$

(3) $\frac{1}{6}(12\hat{i} + 12\hat{j} + \hat{k})$ (4) $\frac{1}{6}(7\hat{i} + 12\hat{j} + \hat{k})$

Ans. (4)

Sol.



Area of $\Delta ABC = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$

$= \frac{1}{2}\left|5\hat{i} + 3\hat{j} + \hat{k}\right| = \frac{1}{2}\sqrt{35}$

volume of tetrahedron

$= \frac{1}{3} \times \text{Base area} \times h = \frac{\sqrt{805}}{6\sqrt{2}}$

$\frac{1}{3} \times \frac{1}{2}\sqrt{35} \times h = \frac{\sqrt{805}}{6\sqrt{2}}$

$h = \sqrt{\frac{23}{2}}$

$AE^2 = AD^2 - DE^2 = \frac{13}{18} \therefore AE = \sqrt{\frac{13}{18}}$

$\overrightarrow{AE} = |AE|.\left(\frac{\hat{i}-5\hat{k}}{\sqrt{26}}\right)$

$= \sqrt{\frac{13}{18}}.\left(\frac{\hat{i}-5\hat{k}}{\sqrt{26}}\right)$

$= \sqrt{\frac{13}{18}}.\left(\frac{\hat{i}-5\hat{k}}{\sqrt{26}}\right) = \frac{\hat{i}-5\hat{k}}{6}$

P.V. of E $= \frac{\hat{i}-5\hat{k}}{6} + \hat{i}+2\hat{j}+\hat{k} = \frac{1}{6}\left(7\hat{i}+12\hat{j}+\hat{k}\right)$

18. If A, B and $(adj(A^{-1}) + adj(B^{-1}))$ are non-singular matrices of same order, then the inverse of $A(adj(A^{-1}) + adj(B^{-1}))^{-1}B$, is equal to

(1) $AB^{-1} + A^{-1}B$ (2) $adj(B^{-1}) + adj(A^{-1})$

(3) $\frac{1}{|AB|}(adj(B) + adj(A))$ (4) $\frac{AB^{-1}}{|A|} + \frac{BA^{-1}}{|B|}$

Ans. **(3)**

Sol. $\left[A\left(adj\left(A^{-1}\right)+adj\left(B^{-1}\right)\right)^{-1}.B\right]^{-1}$

$B^{-1}.\left(adj\left(A^{-1}\right)+adj\left(B^{-1}\right)\right).A^{-1}$

$B^{-1}adj\left(A^{-1}\right)A^{-1}+B^{-1}\left(adj\left(B^{-1}\right)\right).A^{-1}$

$B^{-1}\left|A^{-1}\right|I+\left|B^{-1}\right|IA^{-1}$

$\frac{B^{-1}}{|A|}+\frac{A^{-1}}{|B|}$

$\Rightarrow \frac{adjB}{|B||A|}+\frac{adjA}{|A||B|}$

$= \frac{1}{|A||B|}(adjB + adjA)$

19. If the system of equations

$(\lambda - 1)x + (\lambda - 4)y + \lambda z = 5$

$\lambda x + (\lambda - 1)y + (\lambda - 4)z = 7$

$(\lambda + 1)x + (\lambda + 2)y - (\lambda + 2)z = 9$

has infinitely many solutions, then $\lambda^2 + \lambda$ is equal to

(1) 10 (2) 12

(3) 6 (4) 20

Ans. **(2)**

Sol. $(\lambda-1)x+(\lambda-4)y+\lambda z=5$

$\lambda x+(\lambda-1)y+(\lambda-4)z=7$

$(\lambda+1)x+(\lambda+2)y-(\lambda+2)z=9$

For infinitely many solutions

$$D=\begin{vmatrix}\lambda-1 & \lambda-4 & \lambda\\ \lambda & \lambda-1 & \lambda-4\\ \lambda+1 & \lambda+2 & -(\lambda+2)\end{vmatrix}=0$$

$(\lambda - 3)(2\lambda + 1) = 0$

$$D_x=\begin{vmatrix}5 & \lambda-4 & \lambda\\ 7 & \lambda-1 & \lambda-4\\ 9 & \lambda+2 & -(\lambda+2)\end{vmatrix}=0$$

$2(3 - \lambda)(23 - 2\lambda) = 0$

$\lambda = 3$

$\therefore \lambda^2 + \lambda = 9 + 3 = 12$

20. One die has two faces marked 1, two faces marked 2, one face marked 3 and one face marked 4. Another die has one face marked 1, two faces marked 2, two faces marked 3 and one face marked 4. The probability of getting the sum of numbers to be 4 or 5, when both the dice are thrown together, is

(1) $\frac{1}{2}$ (2) $\frac{3}{5}$

(3) $\frac{2}{3}$ (4) $\frac{4}{9}$

Ans. **(1)**

Sol. a = number or dice 1

b = number on dice 2

(a,b) = (1,3), (3,1), (2,2), (2,3), (3,2), (1,4), (4,1)

Required probability

$= \frac{2}{6}\times\frac{2}{6}+\frac{1}{6}\times\frac{1}{6}+\frac{2}{6}\times\frac{2}{6}+\frac{2}{6}\times\frac{2}{6}+\frac{1}{6}\times\frac{2}{6}+\frac{2}{6}\times\frac{1}{6}+\frac{1}{6}\times\frac{2}{6}$

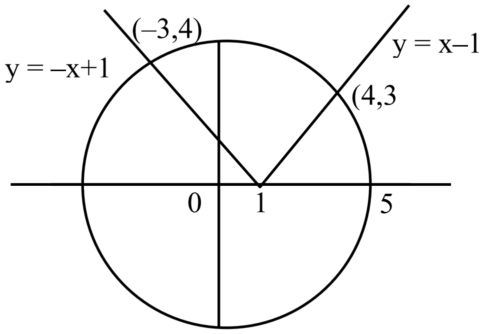
$= \frac{18}{36} = \frac{1}{2}$

SECTION-B

21. If the area of the larger portion bounded between the curves $x^2 + y^2 = 25$ and $y = |x - 1|$ is $\frac{1}{4}(b\pi + c)$, $b, c \in \mathbb{N}$, then $b + c$ is equal to ________

Ans. (77)

Sol.



$x^2 + y^2 = 5$

$x^2 + (x-1)^2 = 25 \Rightarrow x = 4$

$x^2 + (-x+1)^2 = 5 \Rightarrow x = -3$

$$A = 25\pi - \int_{-3}^{4}\sqrt{25-x^2}\,dx + \frac{1}{2}\times 4\times 4 + \frac{1}{2}\times 3\times 3$$

$$A = 25\pi + \frac{25}{2} - \left[\frac{x}{2}\sqrt{25-x^2} + \frac{25}{2}\sin^{-1}\frac{x}{5}\right]_{-3}^{4}$$

$$A = 25\pi + \frac{25}{2} - \left[6 + \frac{25}{2}\sin^{-1}\frac{4}{5} + 6 + \frac{25}{2}\sin^{-1}\frac{3}{5}\right]$$

$$A = 25\pi + \frac{1}{2} - \frac{25}{2}.\frac{\pi}{2}$$

$$A = \frac{75\pi}{4} + \frac{1}{2}$$

$$A = \frac{1}{4}(75\pi + 2)$$

$b = 75$, $c = 2$

$b + c = 75 + 2 = 77$

22. The sum of all rational terms in the expansion of $(1 + 2^{1/3} + 3^{1/2})^6$ is equal to ________

Ans. (612)

Sol. $\left(1 + 2^{\frac{1}{3}} + 3^{\frac{1}{2}}\right)^6$

$$= \frac{\underline{|6}}{\underline{|r_1}\,\underline{|r_2}\,\underline{|r_3}}(1)^{r_1}(2)^{\frac{r_2}{3}}(3)^{\frac{r_3}{2}}$$

r_1	r_2	r_3
6	0	0
4	0	2
2	0	4
0	0	6
3	3	0
1	3	2
0	6	0

$$\text{sum} = \frac{\underline{|6}}{\underline{|6}\,\underline{|0}\,\underline{|0}} + \frac{\underline{|6}}{\underline{|4}\,\underline{|0}\,\underline{|2}}(3) + \frac{\underline{|6}}{\underline{|2}\,\underline{|0}\,\underline{|4}}(3)^2 + \frac{\underline{|6}}{\underline{|0}\,\underline{|0}\,\underline{|6}}(3)^3$$

$$+ \frac{\underline{|6}}{\underline{|3}\,\underline{|3}\,\underline{|0}}(2) + \frac{\underline{|6}}{\underline{|1}\,\underline{|3}\,\underline{|2}}(2)^1(3)^1 + \frac{\underline{|6}}{\underline{|0}\,\underline{|6}\,\underline{|0}}(2)^2$$

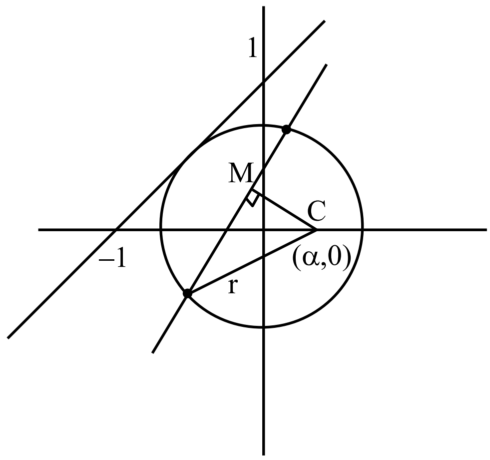
$= 1 + 45 + 135 + 27 + 40 + 360 + 4 = 612$

23. Let the circle C touch the line $x - y + 1 = 0$, have the centre on the positive x-axis, and cut off a chord of length $\frac{4}{\sqrt{13}}$ along the line $-3x + 2y = 1$.

Let H be the hyperbola $\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1$, whose one of the foci is the centre of C and the length of the transverse axis is the diameter of C. Then $2\alpha^2 + 3\beta^2$ is equal to ________

Ans. (19)

Sol.



$x - y + 1 = 0$

$p = r$

$$\left|\frac{\alpha - 0 + 1}{\sqrt{2}}\right| = r \Rightarrow (\alpha + 1)^2 = 2r^2 \;....\;(1)$$

now $\left(\dfrac{-3\alpha+0-1}{\sqrt{9+4}}\right)^2+\left(\dfrac{2}{\sqrt{13}}\right)^2=r^2$

$\Rightarrow (3\alpha + 1)^2 + 4 = 13\, r^2$ ......(2)

(1) & (2) $\Rightarrow (3\alpha + 1)^2 + 4 = 13\ \dfrac{(\alpha+1)^2}{2}$

$\Rightarrow 18\alpha^2 + 12\alpha + 2 + 8 = 13\alpha^2 + 26\alpha + 13$

$\Rightarrow 5\alpha^2 - 14\alpha - 3 = 0$

$\Rightarrow 5\alpha^2 - 15\alpha + \alpha - 3 = 0$

$\Rightarrow 5\alpha^2 - 15\alpha + \alpha - 3 = 0$

$\Rightarrow \alpha = \dfrac{-1}{5}, 3$

$\therefore\ r = 2\sqrt{2}$

How $\alpha e = 3$ and $2\alpha = 4\sqrt{2}$

$\alpha^2 e^2 = 9\ \Rightarrow \alpha = 2\sqrt{2}\ \Rightarrow \alpha^2 = 8$

$\alpha^2\left(1+\dfrac{\beta^2}{\alpha^2}\right)=9$

$\alpha^2 + \beta^2 = 9$

$\therefore \beta^2 = 1$

$\therefore 2\alpha^2 + 3\beta^2 = 2\,(8) + 3(1) = 19$

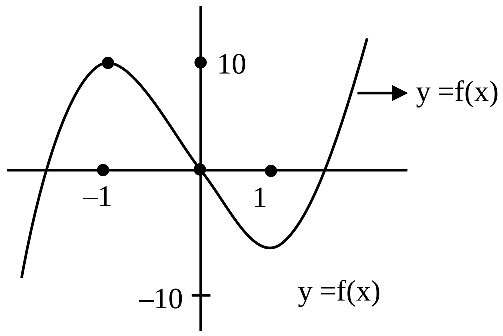
24. If the set of all values of a, for which the equation $5x^3 - 15x - a = 0$ has three distinct real roots, is the interval (α, β), then $\beta - 2\alpha$ is equal to ________

Ans. (30)

Sol. $5x^3 - 15x - a = 0$

$f(x) = 5x^3 - 15x$

$f'(x) = 15x^2 - 15 = 15(x-1)(x + 1)$



$a \in (-10, 10)$

$\alpha = -10,\ \beta = 10$

$\beta - 2\alpha = 10 + 20 = 30$

25. If the equation $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$ has equal roots, where $a + c = 15$ and $b = \dfrac{36}{5}$, then $a^2 + c^2$ is equal to ________

Ans. (117)

Sol. $a(b - c)\, x^2 + b\,(c - a)\, x + c(a - b) = 0$

$x = 1$ is root $\therefore$ other root is 1

$\alpha + \beta = -\dfrac{b(c-a)}{a(b-c)} = 2$

$\Rightarrow -bc + ab = 2ab - 2ac$

$\Rightarrow 2ac = ab + bc$

$\Rightarrow 2ac = b(a + c)$

$\Rightarrow 2ac = 15b$ ... (1)

$\Rightarrow 2ac = 15\left(\dfrac{36}{5}\right) = 108$

$\Rightarrow ac = 54$

$a + c = 15$

$a^2 + c^2 + 2ac = 225$

$a^2 + c^2 = 225 - 108 = 117$

JEE–MAIN EXAMINATION – JANUARY 2025

(HELD ON THURSDAY 23rd JANUARY 2025) **TIME : 3:00 PM TO 6:00 PM**

MATHEMATICS — TEST PAPER WITH SOLUTION

SECTION-A

1. If in the expansion of $(1 + x)^p (1 - x)^q$, the coefficients of x and x^2 are 1 and –2, respectively, then $p^2 + q^2$ is equal to :

(1) 8 (2) 18

(3) 13 (4) 20

Ans. (3)

Sol. $(1+x)^p(1-x)^q = \left({}^pC_0 + {}^pC_1x + {}^pC_2x^2 + \ldots\right)\left({}^qC_0 - {}^qC_1x + {}^qC_2x^2 + \ldots\right)$

coff of $x \equiv {}^pC_0\,{}^qC_1 + {}^pC_1\,{}^qC_0 = 1$

$p - q = 1$

coff of $x^2 \equiv {}^pC_0\,{}^qC_2 - {}^pC_1\,{}^qC_1 + {}^pC_2\,{}^qC_0 = -2$

$\frac{q(q-1)}{2} - pq + \frac{p(p-1)}{2} = -2$

$q^2 - q - 2pq + p^2 - p = -4$

$1 - (p + q) = -4$

$p + q = 5$

$p = 3$

$q = 2$

so $p^2 + q^2 = 13$

2. Let $A = \{(x, y) \in \mathbf{R} \times \mathbf{R} : |x + y| \geq 3\}$ and $B = \{(x, y) \in \mathbf{R} \times \mathbf{R} : |x| + |y| \leq 3\}$.

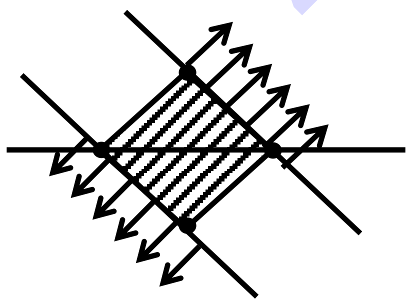
If $C = \{(x, y) \in \mathbf{A} \cap \mathbf{B} : x = 0 \text{ or } y = 0\}$, then $\sum_{(x,y)\in C} |x+y|$ is :

(1) 15 (2) 18

(3) 24 (4) 12

Ans. (4)

Sol.



$C = \{(3,0), (-3,0), (0,3), (0,-3)\}$

$\Sigma|x + y| = 12$

3. The system of equations

$x + y + z = 6,$

$x + 2y + 5z = 9,$

$x + 5y + \lambda z = \mu,$

has no solution if

(1) $\lambda = 17, \mu \neq 18$

(2) $\lambda \neq 17, \mu \neq 18$

(3) $\lambda = 15, \mu \neq 17$

(4) $\lambda = 17, \mu = 18$

Ans. (1)

Sol. $D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 1 & 5 & \lambda \end{vmatrix} = 0$

$\lambda = 17$

$D_z = \begin{vmatrix} 1 & 1 & 6 \\ 1 & 2 & 9 \\ 1 & 5 & \mu \end{vmatrix} \neq 0$

$\mu \neq 18$

4. Let $\int x^3 \sin x\,dx = g(x) + C$, where C is the constant of integration.

If $8\left(g\left(\frac{\pi}{2}\right) + g'\left(\frac{\pi}{2}\right)\right) = \alpha\pi^3 + \beta\pi^2 + \gamma,\ \alpha, \beta, \gamma \in Z,$

Then $\alpha + \beta - \gamma$ equals :

(1) 55 (2) 47

(3) 48 (4) 62

Ans. (1)

Sol. $\int x^3 \sin x\,dx = -x^3\cos x + \int 3x^2 \cos x\,dx$

$= -x^3\cos x + 3x^2 \sin x - \int 6x \sin x\,dx$

$= -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + c$

So $g(x) = -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x$

$g\left(\frac{\pi}{2}\right) = \frac{3\pi^2}{4} - 6$

$g'(x) = -3x^2\cos x + x^3\sin x + 6\cos x - 6\cos x$

$g'\left(\frac{\pi}{2}\right) = \frac{\pi^3}{8}$

$8\left(g\left(\frac{\pi}{2}\right) + g'\left(\frac{\pi}{2}\right)\right) = \pi^3 + 6\pi^2 - 48$

So $\alpha + \beta - \gamma = 55$

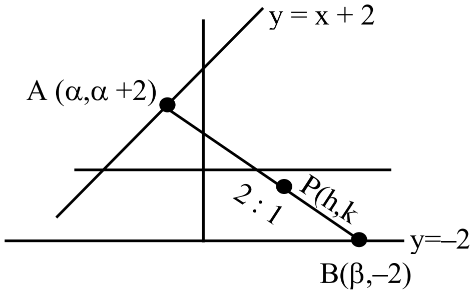
5. A rod of length eight units moves such that its ends A and B always lie on the lines $x - y + 2 = 0$ and $y + 2 = 0$, respectively. If the locus of the point P, that divides the rod AB internally in the ratio 2 : 1 is $9(x^2 + \alpha y^2 + \beta xy + \gamma x + 28y) - 76 = 0$, then $\alpha - \beta - \gamma$ is equal to :

(1) 24 (2) 23

(3) 21 (4) 22

Ans. (2)

Sol.



$$h = \frac{3\beta + \alpha}{3}$$

$$k = \frac{-4 + \alpha + 2}{3}$$

$\alpha = 3k + 2$

$2\beta = 3h - a = 3h - 3k - 2$

so AB = 8

$$(\alpha - \beta)^2 + (\alpha + 4)^2 = 64$$

$$\left(3k + 2 - \left(\frac{3h - 3k - 2}{2}\right)\right)^2 + (3k + 2 + 4)^2 = 64$$

$$\frac{(9k - 3h + 6)^2}{4} + (3k + 6)^2 = 64$$

$$9\left[(3k - h + 2)^2 + 4(k + 2)^2\right] = 64 \times 4$$

$9(x^2 + 13y^2 - 6xy - 4x + 28y) = 76$

$\alpha - \beta - \gamma = 13 + 6 + 4 = 23$

6. The distance of the line $\frac{x-2}{2} = \frac{y-6}{3} = \frac{z-3}{4}$ from the point (1, 4, 0) along the line $\frac{x}{1} = \frac{y-2}{2} = \frac{z+3}{3}$ is :

(1) $\sqrt{17}$ (2) $\sqrt{14}$

(3) $\sqrt{15}$ (4) $\sqrt{13}$

Ans. (2)

Sol. Let the parallel line is

$$\frac{x-1}{1} = \frac{y-4}{2} = \frac{z-0}{3}$$

so their point of intersection is

$(\lambda + 1, 2\lambda + 4\ 3\lambda) = (2t + 2, 3t + 6, 4t + 3)$

$\lambda = 2t + 1$

$2\lambda + 4 = 3t + 6 \Rightarrow t = 0$

so POI is (2,6,3)

so distance $= \sqrt{(2-1)^2 + (6-4)^2 + (3-0)^2} = \sqrt{14}$

7. Let the point A divide the line segment joining the points P(–1, –1, 2) and Q(5, 5,10) internally in the ratio r : 1 (r > 0). If O is the origin and $\left(\overrightarrow{OQ}.\overrightarrow{OA}\right) - \frac{1}{5}\left|\overrightarrow{OP} \times \overrightarrow{OA}\right|^2 = 10$, then the value of r is :

(1) 14 (2) 3

(3) $\sqrt{7}$ (4) 7

Ans. (4)

Sol. $A \equiv \left(\frac{5r-1}{r+1}, \frac{5r-1}{r+1}, \frac{10r+2}{r+1}\right)$

$$\left(\overrightarrow{OQ}.\overrightarrow{OA}\right) - \frac{\left|\overrightarrow{OP} \times \overrightarrow{OA}\right|^2}{5} = 10 \quad ....(1)$$

$$\overrightarrow{OQ}.\overrightarrow{OA} = \frac{5}{r+1}(30r + 2)$$

$$\left|\overrightarrow{OP} \times \overrightarrow{OA}\right|^2 = \frac{r^2}{(r+1)^2}(800)$$

so by equation (1)

$$\frac{10}{r+1}(15r + 1) - \frac{1}{5}\frac{r^2(800)}{(r+1)^2} = 10$$

$2r^2 - 14r = 0$

$r = 7, r \neq 0$

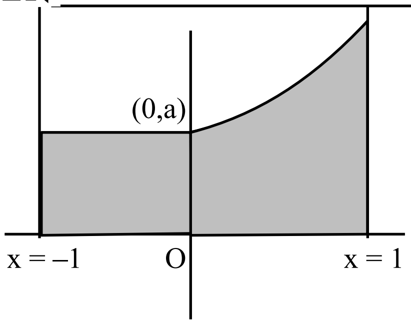
8. If the area of the region $\{(x, y) : -1 \le x \le 1, 0 \le y \le a + e^{|x|} - e^{-x}, a > 0\}$ is $\frac{e^2 + 8e + 1}{e}$, then the value of a is :

(1) 7 (2) 6

(3) 8 (4) 5

Ans. (4)

Sol.



required area is $a+\int_0^1\left(a+e^x-e^{-x}\right)dx$

$a+\left[a+e^x+e^{-x}\right]_0^1$

$2a+e-1+e^{-1}-1=e+8+\frac{1}{e}$

$2a=10 \Rightarrow a=5$

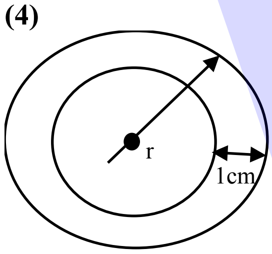
9. A spherical chocolate ball has a layer of ice-cream of uniform thickness around it. When the thickness of the ice-cream layer is 1 cm, the ice-cream melts at the rate of 81 cm^3/min and the thickness of the ice-cream layer decreases at the rate of $\frac{1}{4\pi}$ cm/min. The surface area (in cm^2) of the chocolate ball (without the ice-cream layer) is :

(1) 225π (2) 128π

(3) 196π (4) 256π

Ans. **(4)**

Sol



$v=\frac{4}{3}\pi r^3$

$\frac{dv}{dt}=4\pi r^2\frac{dr}{dt}$

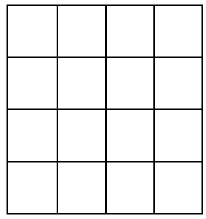
$81=4\pi r^2\times\frac{1}{4\pi}$

$r^2=81$

$r=9$

surface area of chocolate $=4\pi(r-1)^2=256\pi$

10. A board has 16 squares as shown in the figure :



Out of these 16 squares, two squares are chosen at random. The probability that they have no side in common is :

(1) $\frac{4}{5}$ (2) $\frac{7}{10}$

(3) $\frac{3}{5}$ (4) $\frac{23}{30}$

Ans. **(1)**

Sol. Total ways for selecting any two squares $= {}^{16}C_2$

$= 120$

Total ways for selecting common side squares

$=\underbrace{3\times4}_{\text{Horizontal side}}+\underbrace{3\times4}_{\text{vertical side}}$

$= 24$

so required probability

$=1-\frac{24}{120}$

$=\frac{4}{5}$

11. Let $x = x(y)$ be the solution of the differential equation $y=\left(x-y\frac{dx}{dy}\right)\sin\left(\frac{x}{y}\right)$, $y>0$ and $x(1)=\frac{\pi}{2}$.

Then $\cos(x(2))$ is equal to :

(1) $1-2(\log_e 2)^2$ (2) $2(\log_e 2)^2-1$

(3) $2(\log_e 2)-1$ (4) $1-2(\log_e 2)$

Ans. **(2)**

Sol. $ydy=(xdy-ydx)\sin\left(\frac{x}{y}\right)$

$\frac{dy}{y}=\left(\frac{xdy-ydx}{y^2}\right)\sin\left(\frac{x}{y}\right)$

$\frac{dy}{y}=\sin\left(\frac{x}{y}\right)d\left(-\frac{x}{y}\right)$

$\Rightarrow \ell ny=\cos\frac{x}{y}+C$

$x(1) = \frac{\pi}{2} \Rightarrow 0 = \cos\frac{\pi}{2} + C \Rightarrow C=0$

$\ell ny = \cos\frac{x}{y}$

but $y = 2 \Rightarrow \cos\frac{x}{2} = \ell n2$

$\cos x = 2\cos^2\frac{x}{2} - 1$

$= 2(\ell n2)^2 - 1$

12. Let the range of the function

$f(x) = 6 + 16 \cos x \cdot \cos\left(\frac{\pi}{3} - x\right) \cdot \cos\left(\frac{\pi}{3} + x\right) \cdot$

$\sin 3x \cdot \cos 6x$, $x \in R$ be $[\alpha, \beta]$. Then the distance of the point (α, β) from the line $3x + 4y + 12 = 0$ is :

(1) 11 (2) 8

(3) 10 (4) 9

Ans. (1)

Sol. $f(x) = 6 + 16\left(\frac{1}{4}\cos 3x\right)\sin 3x.\cos 6x$

$= 6 + 4 \cos 3x \sin 3x \cos 6x$

$= 6 + \sin 12x$

Range of f(x) is [5, 7]

$(\alpha, \beta) \equiv (5, 7)$

distance $= \left|\frac{15+28+12}{5}\right| = 11$

13. Let the shortest distance from (a, 0), a > 0, to the parabola $y^2 = 4x$ be 4. Then the equation of the circle passing through the point (a, 0) and the focus of the parabola, and having its centre on the axis of the parabola is:

(1) $x^2 + y^2 - 6x + 5 = 0$

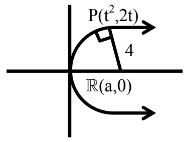
(2) $x^2 + y^2 - 4x + 3 = 0$

(3) $x^2 + y^2 - 10x + 9 = 0$

(4) $x^2 + y^2 - 8x + 7 = 0$

Ans. (1)

Sol. Normal at P



$y + tx = 2t + t^3$

↑

(a, 0)

$at = 2t + t^3$

$a = 2 + t^2$

$\mathbb{R}\ (2 + t^2, 0)$

$P\mathbb{R} = 4 \Rightarrow 4 + 4t^2 = 16$

$4t^2 = 12 \Rightarrow t^2 = 3$

$a = 5 \quad \mathbb{R}\ (5, 0)$

Focus (1, 0)

(1, 0) & (5, 0) will be tha end pts. of diameter

$\Rightarrow$ Eg^n of circle is

$(x - 1)(x - 5) + y^2 = 0$

$x^2 + y^2 - 6x + 5 = 0$

14. Let $X = R \times R$. Define a relation R on X as:

$(a_1, b_1)\ R\ (a_2, b_2) \Leftrightarrow b_1 = b_2$.

Statement-I: R is an equivalence relation.

Statement-II: For some $(a, b) \in X$, the set $S = \{(x, y) \in X : (x, y)\ R\ (a, b)\}$ represents a line parallel to $y = x$.

In the light of the above statements, choose the **correct** answer from the options given below:

(1) Both **Statement-I** and **Statement-II** are false.

(2) **Statement-I** is true but **Statement-II** is false.

(3) Both **Statement-I** and **Statement-II** are true.

(4) **Statement-I** is false but **Statement-II** is true.

Ans. (2)

Sol. Statement – I :

Reflexive : $(a_1, b)\ R(a_1, b_1) \Rightarrow b_1 = b_1$ True

Symmetric : $(a_1, b_1)\ R\ (a_2, b_2) \Rightarrow b_1 = b_2$, $(a_2, b_2)\ R\ (a_1, b_1) \Rightarrow b_2 = b_1$ } True

Transitive : $(a_1, b_1)\ R\ (a_2, b_2) \Rightarrow b_1 = b_2$ & $(a_2, b_2)\ R\ (a_3, b_3)\ b_2 = b_3$ } $b_1 = b_3$

$\Rightarrow (a_1, b_1)\ R\ (a_3, b_3) \Rightarrow$ True

Hence Relation R is an equivence relation Statement-I is true.

For statement – II $\Rightarrow y = b$ so False

15. The length of the chord of the ellipse $\frac{x^2}{4}+\frac{y^2}{2}=1$, whose mid-point is $\left(1,\frac{1}{2}\right)$, is:

(1) $\frac{2}{3}\sqrt{15}$ (2) $\frac{5}{3}\sqrt{15}$

(3) $\frac{1}{3}\sqrt{15}$ (4) $\sqrt{15}$

Ans. (1)

Sol. $T = S_1$

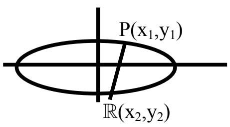
$\frac{x.1}{4}+\frac{y.\frac{1}{2}}{2}=\frac{1}{4}+\frac{1}{8}$

$x + y = \frac{3}{2}$

solve with ellipse

$P_{\mathbb{R}} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$= \sqrt{2}|x_2-x_1|$



$y_2 = \frac{3}{2} - x_2$

$y_1 = \frac{3}{2} - x_1$

$y_2 - y_1 = x_2 - x_1$

$x^2 + 2y^2 = 4$

$x^2 + 2\left(\frac{3}{2}-x\right)^2 = 4$

$6x^2 - 12x + 1 = 0$

$x_1 + x_2 = 2$

$x_1x_2 = 1/6$

$|x_2-x_1| = \sqrt{(x_2+x_1)^2-4x_1x_2}$

$= \sqrt{4-4/6}$

$PR = \sqrt{2}.2.\frac{\sqrt{5}}{\sqrt{2}\sqrt{3}} = \frac{2}{3}\sqrt{15}$

$= 2\sqrt{\frac{5}{6}}$

16. Let $A = [a_{ij}]$ be a 3×3 matrix such that $A\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\1\end{bmatrix}$, $A\begin{bmatrix}4\\1\\3\end{bmatrix}=\begin{bmatrix}0\\1\\0\end{bmatrix}$ and $A\begin{bmatrix}2\\1\\2\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}$, then a_{23} equals:

(1) –1 (2) 0

(3) 2 (4) 1

Ans. (1)

Sol. Let $A = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$

$A\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\1\end{bmatrix} \Rightarrow \begin{bmatrix}a_{12}\\a_{22}\\a_{32}\end{bmatrix}=\begin{bmatrix}0\\1\\0\end{bmatrix} \Rightarrow \begin{matrix}a_{22}=0; a_{12}=0\\ a_{32}=1\end{matrix}$

$A\begin{bmatrix}4\\1\\3\end{bmatrix}=\begin{bmatrix}0\\1\\0\end{bmatrix} \Rightarrow \begin{matrix}4a_{11}+a_{12}+3a_{13}=0\\ 4a_{21}+a_{22}+3a_{23}=1 \Rightarrow 4a_{21}+3a_{23}=1\\ 4a_{31}+a_{32}+3a_{33}=0\end{matrix}$

$A\begin{bmatrix}2\\1\\2\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix} \Rightarrow \begin{matrix}2a_{11}+a_{12}+2a_{13}=1\\ 2a_{21}+a_{22}+2a_{23}=0 \Rightarrow a_{21}+a_{23}=0\\ 2a_{31}+a_{32}+2a_{33}=0\end{matrix}$

$-4a_{23} + 3a_{23} = 1 \Rightarrow a_{23} = -1$

17. The number of complex numbers z, satisfying |z| = 1 and $\left|\frac{z}{\bar{z}}+\frac{\bar{z}}{z}\right|=1$, is :

(1) 6 (2) 4

(3) 10 (4) 8

Ans. (4)

Sol. $z = e^{i\theta}$

$\frac{z}{\bar{z}} = e^{i2\theta}$

$\left|\frac{z}{\bar{z}}+\frac{\bar{z}}{z}\right|=1 \Rightarrow \left|e^{i2\theta}+e^{-i2\theta}\right|=1 \Rightarrow |\cos 2\theta| = \frac{1}{2}$

8 solution

18. If the square of the shortest distance between the lines $\frac{x-2}{1}=\frac{y-1}{2}=\frac{z+3}{-3}$ and $\frac{x+1}{2}=\frac{y+3}{4}=\frac{z+5}{-5}$ is $\frac{m}{n}$, where m, n are coprime numbers, then m + n is equal to:

(1) 6 (2) 9

(3) 21 (4) 14

Ans. (2)

Sol. $\vec{a} = (2,1,-3)$

$\vec{b} = (-1,-3,-5)$

$$\vec{p}\times\vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix}$$

$= 2\hat{i} - \hat{j}$

$\vec{b} - \vec{a} = -3\hat{i} - 4\hat{j} - 2\hat{k}$

$$S_d = \frac{\left|(\vec{b}-\vec{a}).(\vec{p}\times\vec{q})\right|}{|\vec{p}\times\vec{q}|}$$

$$= \frac{2}{\sqrt{5}}$$

$$(S_d)^2 = \frac{4}{5}$$

$m = 4, n = 5 \Rightarrow m + n = 9$

19. If $I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} dx$,

then $\int_0^{21} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx$ equals:

(1) $\frac{\pi^2}{16}$ (2) $\frac{\pi^2}{4}$

(3) $\frac{\pi^2}{8}$ (4) $\frac{\pi^2}{12}$

Ans. **(1)**

Sol. For I

Apply king (P– 5) and add

$$2I = \int_0^{\pi/2} dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

$$I_2 = \int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx$$

Apply king and add

$$I_2 = \frac{\pi}{4}\int_0^{\pi/2} \frac{\tan x \sec^2 x dx}{\tan^4 x + 1}$$

put $\tan^2 x = t$

$$\frac{\pi}{8}\int_0^{\infty} \frac{dt}{t^2+1}$$

$$= \frac{\pi}{8}.\frac{\pi}{2} = \frac{\pi^2}{16}$$

20. $\lim_{x\to\infty} \frac{(2x^2-3x+5)(3x-1)^{\frac{x}{2}}}{(3x^2+5x+4)\sqrt{(3x+2)^x}}$ is equal to:

(1) $\frac{2}{\sqrt{3e}}$ (2) $\frac{2e}{\sqrt{3}}$

(3) $\frac{2e}{3}$ (4) $\frac{2}{3\sqrt{e}}$

Ans. **(4)**

Sol. $$\lim_{x\to\infty} \frac{\left(2-\frac{3}{x}+\frac{5}{x^2}\right)\left(1-\frac{1}{3x}\right)^{x/2}}{\left(3+\frac{5}{x}+\frac{4}{x^2}\right)\left(1+\frac{2}{3x}\right)^{x/2}}$$

$$= \lim_{x\to\infty} \frac{2}{3}.\frac{e^{\frac{x}{2}\left(1-\frac{1}{3x}-1\right)}}{e^{\frac{x}{2}\left(1+\frac{2}{3x}-1\right)}}$$

$$= \frac{2}{3}.\frac{e^{-\frac{1}{6}}}{e^{1/3}} = \frac{2}{3}e^{-\frac{1}{2}}$$

SECTION-B

21. The number of ways, 5 boys and 4 girls can sit in a row so that either all the boys sit together or no two boys sit together, is ________.

Ans. **(17280)**

Sol. A : number of ways that all boys sit together = 5! × 5!

B : number of ways if no 2 boys sit together = 4! × 5!

$A \cap B = \phi$

Required no. of ways = 5! × 5! + 4! × 5! = 17280

22. Let α, β be the roots of the equation $x^2 - ax - b = 0$ with $\text{Im}(\alpha) < \text{Im}(\beta)$. Let $P_n = \alpha^n - \beta^n$. If $P_3 = -5\sqrt{7}i$, $P_4 = -3\sqrt{7}i$, $P_5 = 11\sqrt{7}i$ and $P_6 = 45\sqrt{7}i$, then $|\alpha^4 + \beta^4|$ is equal to ________.

Ans. **(31)**

Sol. $\alpha + \beta = a \quad \alpha\beta = -b$

$P_6 = aP_5 + bP_4$

$45\sqrt{7}i = a \times 11\sqrt{7}i + b(-3\sqrt{7})i$

$45 = 11a - 3b$ ...(1)

and

$P_5 = aP_4 + bP_3$

$11\sqrt{7}i = a\left(-3\sqrt{7}i\right) + b\left(-5\sqrt{7}i\right)$

$11 = -3a - 5b$ ...(2)

$a = 3, b = -4$

$$\left|\alpha^4 + \beta^4\right| = \sqrt{\left(\alpha^4 - \beta^4\right)^2 + 4\alpha^4\beta^4}$$

$$= \sqrt{-63 + 4.4^4}$$

$$= \sqrt{-63 + 1024} = \sqrt{961} = 31$$

23. The focus of the parabola $y^2 = 4x + 16$ is the centre of the circle C of radius 5. If the values of λ, for which C passes through the point of intersection of the lines $3x - y = 0$ and $x + \lambda y = 4$, are λ_1 and λ_2, $\lambda_1 < \lambda_2$, then $12\lambda_1 + 29\lambda_2$ is equal to _______.

Ans. (15)

Sol. $y^2 = 4(x + 4)$

Equation of circle

$(x + 3)^2 + y^2 = 25$

Passes through the point of intersection of two lines $3x - y = 0$ and $x + \lambda y = 4$

$\left(\frac{4}{3\lambda+1}, \frac{12}{3\lambda+1}\right)$, we get

$\lambda = -\frac{7}{6}, 1$

$12\lambda_1 + 29\lambda_2$

$-14 + 29 = 15$

24. The variance of the numbers 8, 21, 34, 47, …, 320, is _________.

Ans. (8788)

Sol. $8 + (n-1)13 = 320$

$13n = 325$

$n = 25$

no. of terms = 25

$$\text{mean} = \frac{\Sigma x_i}{n} = \frac{8 + 21 + ... + 320}{25} = \frac{\frac{25}{2}(8+320)}{25}$$

$$\text{variance } \sigma^2 = \frac{\Sigma x_i^2}{n} - (\text{mean})^2$$

$$= \frac{8^2 + 21^2 + ... + 320^2}{13} - (164)^2$$

$= 8788$

25. The roots of the quadratic equation $3x^2 - px + q = 0$ are 10^{th} and 11^{th} terms of an arithmetic progression with common difference $\frac{3}{2}$. If the sum of the first 11 terms of this arithmetic progression is 88, then $q - 2q$ is equal to _________.

Ans. (474)

Sol. $S_{11} = \frac{11}{2}(2a + 10d) = 88$

$a + 5d = 8$

$a = 8 - 5 \times \frac{3}{2} = \frac{1}{2}$

Roots are

$T_{10} = a + 9d = \frac{1}{2} + 9 \times \frac{3}{2} = 14$

$T_{11} = a + 10d = \frac{1}{2} + 10 \times \frac{3}{2} = \frac{31}{2}$

$\frac{p}{3} = T_{10} + T_{11} = 14 + \frac{31}{2} = \frac{59}{2}$

$p = \frac{177}{2}$

$\frac{q}{3} = T_{10} \times T_{11} = 7 \times 31 = 217$

$q = 651$

$q - 2p$

$= 651 - 177$

$= 474$

JEE–MAIN EXAMINATION – JANUARY 2025

(HELD ON FRIDAY 24th JANUARY 2025) **TIME : 9:00 AM TO 12:00 PM**

MATHEMATICS — TEST PAPER WITH SOLUTION

SECTION-A

1. Let $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$, $\vec{b}=3\hat{i}+\hat{j}-\hat{k}$ and $\vec{c}$ be three vectors such that $\vec{c}$ is coplanar with $\vec{a}$ and $\vec{b}$. If the vector $\vec{c}$ is perpendicular to $\vec{b}$ and $\vec{a}.\vec{c}=5$, then $|\vec{c}|$ is equal to

(1) $\frac{1}{3\sqrt{2}}$ (2) 18

(3) 16 (4) $\sqrt{\frac{11}{6}}$

Ans. **(4)**

Sol. $\vec{c}=\lambda(\vec{b}\times(\vec{a}\times\vec{b}))$

$=\lambda((\vec{b}.\vec{b})\vec{a}-(\vec{a}.\vec{b})\vec{b})$

$=\lambda(11\vec{a}-2\vec{b})=\lambda(11i+22j+33k-6i-2j+2k)$

$=\lambda(5i+20j+35k)$

$=5\lambda(5i+4j+7k)$

$=$ Given $\vec{c}.\vec{a}=5$

$=5\lambda(1+8+21)=5 \quad =\lambda=\frac{1}{30}$

$\Rightarrow \vec{c}=\frac{1}{6}(i+4j+7k)$

$|\vec{c}|=\frac{\sqrt{1+16+49}}{6}=\sqrt{\frac{11}{6}}$

2. In $I(m, n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$, m, n > 0, then I(9, 14) + I(10, 13) is

(1) I(9, 1) (2) I(19, 27)

(3) I(1, 13) (4) I(9, 13)

Ans. **(4)**

Sol. $I(m,m)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$

Let $x=\sin^2\theta$ $dx=2\sin\theta\cos\theta d\theta$

$I(m,n)=2\int_0^{\pi/2}(\sin\theta)^{2m-1}(\cos\theta)^{2n-1}d\theta$

$I(9,14)+I(10,13)=2\int_0^{\pi/2}(\sin\theta)^{17}(\cos\theta)^{27}d\theta$

$+2\int_0^{\pi/2}(\sin\theta)^{19}(\cos\theta)^{25}d\theta$

$=2\int_0^{\pi/2}(\sin\theta)^{17}(\cos\theta)^{25}d\theta$

$=I(9, 13)$

3. Let $f:\mathbb{R}-\{0\}\to\mathbb{R}$ be a function such that $f(x)-6f\left(\frac{1}{x}\right)=\frac{35}{3x}-\frac{5}{2}$. If the $\lim_{x\to0}\left(\frac{1}{\alpha x}+f(x)\right)=\beta$; $\alpha, \beta\in\mathbb{R}$, then $\alpha+2\beta$ is equal to

(1) 3 (2) 5

(3) 4 (4) 6

Ans. **(3)**

Sol. $F(x)-6f(1/x)=\frac{35}{3x}-\frac{5}{2}$ ……(1)

Replace $x\to\frac{1}{x}$

$F(1/x)-6(x)=\frac{35x}{3}-\frac{5}{2}$ ……(2)

Using (1) & (2)

$f(x)=-2x-\frac{1}{3x}+\frac{1}{2}$

$B=\lim_{x\to0}\left(\frac{1}{\alpha x}+f(x)\right)$

$=\lim_{x\to0}\left(\frac{1}{\alpha x}-2x-\frac{1}{3x}+\frac{1}{2}\right)$

$\alpha=3$, $B=½$

So, $\alpha+2B=3+1=4$

4. Let $S_n=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\ldots$ upto n terms. If the sum of the first six terms of an A.P. with first term –p and common difference p is $\sqrt{2026S_{2025}}$, then the absolute difference between 20th and 15th terms of the A.P. is

(1) 25 (2) 90

(3) 20 (4) 45

Ans. **(1)**

Sol. $Sn=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}$ …. N terms

$S_{2025}=\sum_{n=1}^{2025}\frac{1}{n(n+1)}=\sum_{n=1}^{2025}\left(\frac{1}{n}-\frac{1}{n+1}\right)$

$=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)\ \ldots\ldots\left(\frac{1}{2025}-\frac{1}{2026}\right)$

$= \dfrac{2025}{2026}$

$\sqrt{2026.S_{2025}} = \sqrt{2025} = 45$

Given : $\dfrac{6}{2}[-2p+(6-1)p]=45$

$9p = 45$

$p = 5$

$|A_{20} - A_{15}| = |-5 + 19 \times 5| - [-5 + 14 \times 5]$

$= |90 - 65|$

$= 25$

5. Let $f(x) = \dfrac{2^{x+2}+16}{2^{2x+1}+2^{x+4}+32}$. Then the value of $8\left(f\left(\dfrac{1}{15}\right)+f\left(\dfrac{2}{15}\right)+\ldots+f\left(\dfrac{59}{15}\right)\right)$ is equal to

(1) 118 (2) 92

(3) 102 (4) 108

Ans. (1)

Sol. $f(x) = \dfrac{42^{x}+16}{2.2^{2x}+16.2^{x}+32}$

$f(x) = \dfrac{2(2^{x}+4)}{2^{2x}+8.2^{x}+16}$

$f(x) = \dfrac{2}{2^{x}+4}$

$f(4-x) = \dfrac{2^{x}}{2(2^{x}+4)}$

$f(x) + f(4-x) = \dfrac{1}{2}$

So, $f\left(\dfrac{1}{15}\right)+f\left(\dfrac{59}{15}\right)=\dfrac{1}{2}$

Similarly = $f\left(\dfrac{29}{15}\right)+f\left(\dfrac{31}{15}\right)=\dfrac{1}{2}$

$f\left(\dfrac{30}{15}\right)=f(2)=\dfrac{2}{2^2+4}=\dfrac{2}{8}=\dfrac{1}{4}$

$\Rightarrow 8\left(29\times\dfrac{1}{2}+\dfrac{1}{4}\right)$

Ans. 118

Option (4)

6. If α and β are the roots of the equation $2z^2 - 3z - 2i = 0$, where $i=\sqrt{-1}$, then $16.\mathrm{Re}\left(\dfrac{\alpha^{19}+\beta^{19}+\alpha^{11}+\beta^{11}}{\alpha^{15}+\beta^{15}}\right).\mathrm{Im}\left(\dfrac{\alpha^{19}+\beta^{19}+\alpha^{11}+\beta^{11}}{\alpha^{15}+\beta^{15}}\right)$ is equal to

(1) 398 (2) 312

(3) 409 (4) 441

Ans. (4)

Sol. $2z^2 - 32 - 2i = 0$

$2\left(z - \dfrac{i}{z}\right) = 3$

$\alpha - \dfrac{i}{\alpha} = \dfrac{3}{2}$

$\Rightarrow \alpha^2 - \dfrac{1}{\alpha^2} - 2i = \dfrac{9}{4}$

$\Rightarrow \alpha^2 - \dfrac{1}{\alpha^2} - 2i = \dfrac{9}{4}$

$\Rightarrow \dfrac{9}{4} + 2i = \alpha^2 - \dfrac{1}{\alpha^2}$

$\Rightarrow \dfrac{81}{16} - 4 + 9i = \alpha^4 + \dfrac{1}{\alpha^4} - 2$

$\Rightarrow \dfrac{49}{16} + 9i = \alpha^4 + \dfrac{1}{\alpha^4}$

Similarly

$\Rightarrow \dfrac{49}{16} + 9i = \beta^4 + \dfrac{1}{\beta^4}$

$\Rightarrow \dfrac{\alpha^{19}+\beta^{19}+\alpha^{11}+\beta^{11}}{\alpha^{15}+\beta^{15}} = \dfrac{\alpha^{15}\left(\alpha^4+\dfrac{1}{\alpha^4}\right)+\beta^{15}\left(\beta^4+\dfrac{1}{\beta^4}\right)}{\alpha^{15}+\beta^{15}}$

$= \dfrac{(\alpha^{15}+\beta^{15})\left(\dfrac{49}{16}+9i\right)}{(\alpha^{15}+\beta^{15})}$

Real $= \dfrac{49}{16}$

Im = 9

Ans. 441

7. $\lim\limits_{x\to 0} \mathrm{cosec}\,x\left(\sqrt{2\cos^2 x+3\cos x}-\sqrt{\cos^2 x+\sin x+4}\right)$ is

(1) 0 (2) $\dfrac{1}{2\sqrt{5}}$

(3) $\dfrac{1}{\sqrt{15}}$ (4) $-\dfrac{1}{2\sqrt{5}}$

Ans. (4)

Sol. $\lim\limits_{x\to 0}\operatorname{cosec}x\left(\sqrt{2\cos^2 x+3\cos x}-\sqrt{\cos^2 x+\sin x+4}\right)$

$$\lim_{x\to 0}\frac{\operatorname{cosec}x\left(\cos^2 x+3\cos x-\sin x-4\right)}{\left(\sqrt{2\cos^2 x+3\cos x}+\sqrt{\cos^2 x+\sin x+4}\right)}$$

$$\lim_{x\to 0}\frac{1}{\sin x}\frac{\left(\cos^2 x+3\cos x-4\right)-\sin x}{\left(\sqrt{2\cos^2 x+3\cos x}+\sqrt{\cos^2 x+\sin x+4}\right)}$$

$$\lim_{x\to 0}\frac{(\cos x+4)(\cos x-1)-\sin x}{\sin x\left(\sqrt{2\cos^2 x+3\cos x}+\sqrt{\cos^2 x+\sin x+4}\right)}$$

$$\lim_{x\to 0}\frac{-2\sin^2\frac{x}{2}(\cos x+4)-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}\left(\sqrt{2\cos^2 x+3\cos x}+\sqrt{\cos^2 x+\sin x+4}\right)}$$

$$\lim_{x\to 0}\frac{-\left(\sin\frac{x}{2}(\cos x+4)+\cos\frac{x}{2}\right)}{\cos\frac{x}{2}\left(\sqrt{2\cos^2 x+3\cos x}+\sqrt{\cos^2 x+\sin x+4}\right)}$$

$$-\frac{1}{2\sqrt{5}}$$

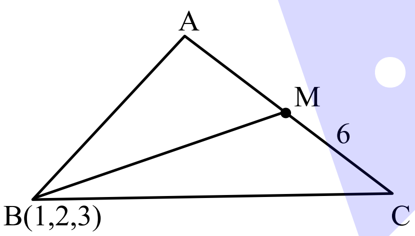
8. Let in a ΔABC, the length of the side AC be 6, the vertex B be (1, 2, 3) and the vertices A, C lie on the line $\frac{x-6}{3}=\frac{y-7}{2}=\frac{z-7}{-2}$. Then the area (in sq. units) of ΔABC is

(1) 42 (2) 21

(3) 56 (4) 17

Ans. **(2)**

Sol.



Let M $(3\lambda+6, 2\lambda+7, -2\lambda+7)$

$\overrightarrow{BM}=(3\lambda+5)\hat{i}+(2\lambda+5)\hat{j}+(-2\lambda+4)\hat{k}$

$\overrightarrow{AC}.\overrightarrow{BM}=0=3(3\lambda+5)+2(2\lambda+5)-2(-2\lambda+4)$

$\overrightarrow{BM}=2\hat{i}+3\hat{j}+6\hat{k}$

$|\overrightarrow{BM}|=7$

Area $=\frac{1}{2}\times 6\times 7=21$

Option (2)

9. Let $y = y(x)$ be the solution of the differential equation $\left(xy-5x^2\sqrt{1+x^2}\right)dx+(1+x^2)dy=0$, $y(0)=0$. Then $y\left(\sqrt{3}\right)$ is equal to

(1) $\frac{5\sqrt{3}}{2}$ (2) $\sqrt{\frac{14}{3}}$

(3) $2\sqrt{2}$ (4) $\sqrt{\frac{15}{2}}$

Ans. **(1)**

Sol. $\left(1+x^2\right)\frac{dy}{dx}+xy=5x^1\sqrt{1+x^2}$

$\frac{dy}{dx}+\frac{xy}{1+x^2}=\frac{5x^2}{\sqrt{1+x^2}}$

$\therefore$ I.F. $=e^{\int\frac{x}{1+x^2}dx}=e^{\frac{\ln\left(1+x^2\right)}{2}}=\sqrt{1+x^2}$

$\therefore\ y\sqrt{1+x^2}=\int\frac{5x^2}{\sqrt{1+x^2}}.\sqrt{1+x^2}dx$

$\therefore\ y\sqrt{1+x^2}=\int\frac{5x^2}{\sqrt{1+x^2}}.\sqrt{1+x^2}dx$

$y\sqrt{1+x^2}=\frac{5x^3}{3}+C$

$\because y(0)=0\Rightarrow 0=0+C\Rightarrow C=0$

$\therefore\ y=\frac{5x^3}{3\sqrt{1+x^2}}$

$y\left(\sqrt{3}\right)=\frac{15\sqrt{3}}{32}=\boxed{\frac{5\sqrt{3}}{2}}$

Option (1)

10. Let the product of the focal distances of the point $\left(\sqrt{3},\frac{1}{2}\right)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $(a>b)$, be $\frac{7}{4}$. Then the absolute difference of the eccentricities of two such ellipses is

(1) $\frac{3-2\sqrt{2}}{3\sqrt{2}}$ (2) $\frac{1-\sqrt{3}}{\sqrt{2}}$

(3) $\frac{3-2\sqrt{2}}{2\sqrt{3}}$ (4) $\frac{1-2\sqrt{2}}{\sqrt{3}}$

Ans. **(3)**

Sol. Product of focal distances = $(a + ex_1)(a - ex_1)$

$= a^2 - e^2x_1^2 = a^2 - e^2(3)$

$= a^2 - 3e^2 = \frac{7}{4} \Rightarrow a^2 = \frac{7}{4} + 3e^2$

$\Rightarrow 4a^2 = 7 + 12e^2$

& $\left(\sqrt{3}, \frac{1}{2}\right)$ lines on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\therefore \frac{3}{a^2} + \frac{1}{4b^2} = 1$

$\frac{3}{a^2} + \frac{1}{4(a^2)(1-e^2)} = 1$

$12(1 - e^2) + 1 = 4a^2(1 - e^2)$

$13 - 12e^2 = (7 + 12e^2)(1 - e^2)$

$\Rightarrow 13 - 12e^2 = 7 - 7e^2 + 12e^2 - 12e^4$

$\Rightarrow 12e^4 - 17e^2 + 6 = 0$

$\therefore e^2 = \frac{17 \pm \sqrt{289 - 288}}{24} = \frac{17 \pm 1}{24} = \frac{3}{4} \& \frac{2}{3}$

$\therefore e = \frac{\sqrt{3}}{2} \& \sqrt{\frac{2}{3}}$

$\therefore$ difference $= = \frac{\sqrt{3}}{2} - \sqrt{\frac{2}{3}} = \frac{3 - 2\sqrt{2}}{2\sqrt{3}}$

Option (3)

11. A and B alternately throw a pair of dice. A wins if he throws a sum of 5 before B throws a sum of 8, and B wins if he throws a sum of 8 before A throws a sum of 5. The probability, that A wins if A makes the first throw, is

(1) $\frac{9}{17}$ (2) $\frac{9}{19}$

(3) $\frac{8}{17}$ (4) $\frac{8}{19}$

Ans. (2)

Sol. $p(S_5) = \frac{1}{9}$

$p(S_5) = \frac{5}{36}$

required prob $= \frac{1}{9} + \frac{8}{9} \cdot \frac{31}{36} \cdot \frac{1}{9} + \left(\frac{8}{9} \cdot \frac{31}{36}\right)^2 \cdot \frac{1}{9} + \ldots\infty$

$= \frac{\frac{1}{9}}{1 - \frac{62}{81}} = \frac{9}{19}$

Option(2)

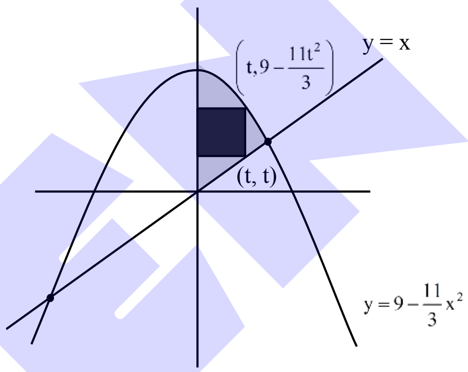
12. Consider the region $R = \left\{(x, y) : x \le y \le 9 - \frac{11}{3}x^2, x \ge 0\right\}$. The area, of the largest rectangle of sides parallel to the coordinate axes and inscribed in R, is :

(1) $\frac{625}{111}$ (2) $\frac{730}{119}$

(3) $\frac{567}{121}$ (4) $\frac{821}{123}$

Ans. (3)

Sol. $t.\left(9 - \frac{11t^2}{3} - t\right)$



$A = 9t - t^2 - \frac{11}{3}t^3$

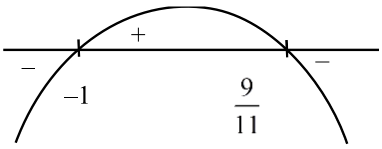
$\frac{dA}{dt} = 9 - 2t - 11t^2$

$\Rightarrow 11t^2 + 2t - 9 = 0$

$11t^2 + 11t - 9t - 9 = 0$

$t = -1 \& t = \frac{9}{11}$

$\therefore \frac{dA}{dt} =$



↓

Maxima at $t = \frac{9}{11}$

$\therefore$ largest area $= \frac{9}{11}\left(9 - \frac{11}{3} \cdot \frac{81}{121} - \frac{9}{11}\right)$

$= \frac{9}{11} \cdot \frac{63}{11} = \frac{567}{121}$

Option (3)

13. The area of the region $\{(x, y) : x^2 + 4x + 2 \le y \le |x + 2|\}$ is equal to

(1) 7 (2) 24/5

(3) 20/3 (4) 5

Ans. (3)

Sol. $x^2 + 4x + 2 \le y \le |x + 2|$

The area bounded between

$y = x^2 + 4x + 2 = (x + 2)^2 - 2$

and $y = |x + 2|$ is same as

area bounded between $y = x^2 - 2$ and $y = |x|$

For P.O.I $|x|^2 - 2 = |x|$

$\Rightarrow |x| = 2 \Rightarrow x = \pm 2$

$\therefore$ Required area $= -\int_{-2}^{2}(x^2 - 2)dx + \int_{-2}^{2}|x|\,dx$

$= -2\int_{0}^{2}(x^2 - 2)dx + 2\int_{0}^{2}x.dx$

$= -2\left[\frac{x^3}{3} - 2x\right]_0^2 + 2\left[\frac{x^2}{2}\right]_0^2$

$= --2\left[\frac{8}{3} - 4\right] + 2\left[\frac{4}{2}\right]$

$= -2 \times \left(\frac{-4}{3}\right) + 4$

$= \frac{20}{3}$

14. For a statistical data $x_1, x_2, \ldots, x_{10}$ of 10 values, a student obtained the mean as 5.5 and $\sum_{i=1}^{10} x_i^2 = 371$. He later found that he had noted two values in the data incorrectly as 4 and 5, instead of the correct values 6 and 8, respectively. The variance of the corrected data is

(1) 7 (2) 4

(3) 9 (4) 5

Ans. (1)

Sol. Mean $\bar{x} = 5.5$

$= \sum_{i=1}^{10} x_i = 5.5 \times 10 = 55$

$= \sum_{i=1}^{10} x_i^2 = 371$

$\left(\sum x_i\right)_{new} = 55 - (4+5) + (6+8) = 60$

$\left(\sum x_i\right)_{new} = 371 - (4^2 + 5^2) + (6^2 + 8^2) = 430$

Variance $\sigma^2 = \frac{\sum x_i^2}{10} - \left(\frac{\sum x_i}{10}\right)^2$

$\sigma^2 = \frac{430}{10} - \left(\frac{60}{10}\right)^2$

$\sigma^2 = 43 - 36$

$\sigma^2 = 7$

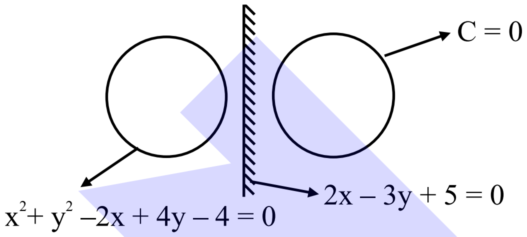
15. Let circle C be the image of $x^2 + y^2 - 2x + 4y - 4 = 0$ in the line $2x - 3y + 5 = 0$ and A be the point on C such that OA is parallel to x-axis and A lies on the right hand side of the centre O of C. If $B(\alpha,\beta)$, with $\beta < 4$, lies on C such that the length of the arc AB is $(1/6)^{th}$ of the perimeter of C, then $\beta - \sqrt{3}\alpha$ is equal to

(1) 3 (2) $3+\sqrt{3}$

(3) $4-\sqrt{3}$ (4) 4

Ans. (4)

Sol.



Centre (1, –2), r = 3

Reflection of (1, –2) about $2x - 3y + 5 = 0$

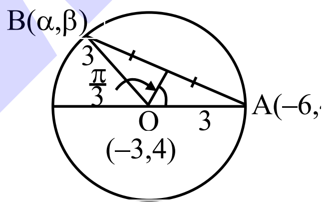
$\frac{x-1}{2} = \frac{y+2}{-3} = \frac{-2(2+6+5)}{13} = -2$

$x = -3, y = 4$

Equation of circle 'C'

$C : (x+3)^2 + (y-4)^2 = 9$

A.T.Q.



$AB = 3\sqrt{3}$

$\ell(\text{arc}AB) = \frac{1}{6} \times 2\pi r$

$r\theta = \frac{1}{6} \times 2\pi r$

$\theta = \frac{\pi}{3}$

$(\alpha + 6)^2 + (\beta - 4)^2 = 27$

$(\alpha + 3)^2 \pm (\beta - 4)^2 = 9$

$(\alpha + 6)^2 - (\alpha + 3)^2 = 18$

$\Rightarrow 6\alpha = -9$

$\Rightarrow \boxed{\alpha = \frac{-3}{2}}, \boxed{\beta = \left(4 - \frac{3\sqrt{3}}{2}\right)}$

$\therefore \beta - \sqrt{3}\alpha$

$\left(4 - \frac{3\sqrt{3}}{2}\right) + \frac{3\sqrt{3}}{2}$

$= 4$

16. For some $n \neq 10$, let the coefficients of the 5^{th}, 6^{th} and 7^{th} terms in the binomial expansion of $(1 + x)^{n+4}$ be in A.P. Then the largest coefficient in the expansion of $(1 + x)^{n+4}$ is :

(1) 70 (2) 35

(3) 20 (4) 10

Ans. **(2)**

Sol. $(1 + x)^{n+4}$

${}^{n+4}C_4, {}^{n+4}C_5, {}^{n+4}C_6, \to$ A.P.

$\Rightarrow 2 \times {}^{n+4}C_5 = {}^{n+4}C_4 + {}^{n+4}C_6$

$\Rightarrow 4 \times {}^{n+4}C_5 = ({}^{n+4}C_4 + {}^{n+4}C_5) + ({}^{n+4}C_5 + {}^{n+4}C_6)$

$\Rightarrow 4 \times {}^{n+4}C_5 = {}^{n+5}C_5 + {}^{n+5}C_6$

$$\Rightarrow 4 \times \frac{(n+4)!}{5!.(n-1)!} = \frac{(n+6)!}{6!.n!}$$

$$\Rightarrow 4 = \frac{(n+6)(n+5)}{6n}$$

$\Rightarrow n^2 + 11n + 30 = 24n$

$\Rightarrow n^2 - 13n + 30 = 0$

$\Rightarrow n = 3, 10$(rejected)

$\because n \neq 10$

$\therefore$ Largest binomial coefficient in expansion of $(1 + x)^7$

$(\because n + 4 = 7)$

is coeff. of middle term

$\Rightarrow {}^7C_4 = {}^7C_3 = 35$

N.T.A. Ans Option (2)

17. The product of all the rational roots of the equation $(x^2 - 9x + 11)^2 - (x - 4)(x - 5) = 3$, is equal to :

(1) 14 (2) 7

(3) 28 (4) 21

Ans. **(1)**

Sol. $(x^2 - 9x + 11)^2 - (x^2 - 9x + 20) = 3$

Let

$\Rightarrow x^2 - 9x = t$

$\Rightarrow t^2 + 22t + 121 - t - 20 - 3 = 0$

$\Rightarrow t^2 + 21t + 98 = 0$

$\Rightarrow (t + 14)(t + 7) = 0$

$\Rightarrow t = -7, -14$

So, $x^2 - 9x = -7, -14$

$x^2 - 9x + 7 = 0$ or $x^2 - 9x + 14 = 0$

$$x = \frac{9 \pm \sqrt{81 - 4(7)}}{2 \times 1} \qquad x = \frac{9 \pm \sqrt{81 - 4(14)}}{2}$$

$$= \frac{9 \pm \sqrt{53}}{2} \qquad = \frac{9 \pm 5}{2}$$

Product of all rational roots $= 7 \times 2 = 14$

Option (1)

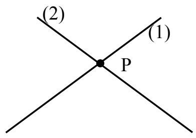
18. Let the line passing through the points $(-1, 2, 1)$ and parallel to the line $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z}{4}$ intersect the line $\frac{x+2}{3} = \frac{y-3}{2} = \frac{z-4}{1}$ at the point P. Then the distance of P from the point $Q(4, -5, 1)$ is :

(1) 5 (2) 10

(3) $5\sqrt{6}$ (4) $5\sqrt{5}$

Ans. **(4)**

Sol. Equation of line through point $(-1, 2, 1)$ is $\to$



$$\Rightarrow \frac{x+1}{2} = \frac{y-2}{3} = \frac{z-1}{4} - (2) = \lambda$$

So, $\begin{bmatrix} x = 2\lambda - 1 \\ y = 3\lambda + 2 \\ z = 4\lambda + 1 \end{bmatrix}$

$$\text{By (1)} \to \frac{x+2}{3} = \frac{y-3}{2} = \frac{z-4}{1} = \mu \text{ (Let)}$$

So, $\begin{bmatrix} x = 3\mu - 2 \\ y = 2\mu + 3 \\ z = \mu + 4 \end{bmatrix}$

For intersection point 'P'

$x = 2\lambda - 1 = 3\mu - 2$

$y = 3\lambda + 2 = 2\mu + 3$ $\begin{bmatrix} \lambda = 1 \\ \mu = 1 \end{bmatrix}$

$z = 4\lambda + 1 = \mu + 4$

So, point $P(x, y, z) = (1, 5, 5)$

& $Q(4, -5, 1)$

$\therefore PQ = \sqrt{9 + 100 + 16}$

$= \sqrt{125} = 5\sqrt{5}$

Option (4)

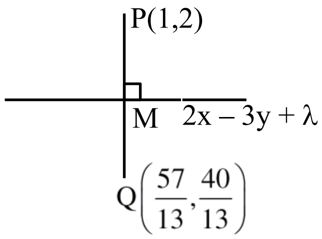
19. Let the lines $3x - 4y - \alpha = 0$, $8x - 11y - 33 = 0$, and $2x - 3y + \lambda = 0$ be concurrent. If the image of the point (1, 2) in the line $2x - 3y + \lambda = 0$ is $\left(\frac{57}{13}, \frac{-40}{13}\right)$, then $|\alpha\lambda|$ is equal to :

(1) 84 (2) 91

(3) 113 (4) 101

Ans. **(2)**

Sol.



$\because PM = QM$

So, $M\left(\dfrac{\frac{57}{13}+1}{2}, \dfrac{\frac{-40}{13}+2}{2}\right)$

$= \left(\frac{35}{13}, \frac{-7}{13}\right)$

$\because$ M lies on the time

$2x - 3y + \lambda = 0$

$2\left(\frac{35}{13}\right) - 3\left(\frac{-7}{13}\right) + \lambda = 0$

$\lambda = -\frac{70}{13} + \frac{21}{13}$

$= \frac{-91}{13} = -7$

$$\begin{vmatrix} 3 & -4 & -\alpha \\ 8 & -11 & -33 \\ 2 & 3 & \lambda \end{vmatrix} = 0$$

$\Rightarrow 3(-11\lambda - 99) + 4(8\lambda + 66) - \alpha(-24 + 22) = 0$

$\Rightarrow 33\lambda - 297 + 32\lambda + 264 + 24\alpha - 22\alpha = 0$

$\Rightarrow -\lambda + 2\alpha - 33 = 0$ .....(1)

$\therefore \lambda = -7$

$-(-7) + 2\alpha - 33 = 0$

$2\alpha = 26$

$\alpha = 13$

$\therefore |\alpha\lambda| = |13 \times (-7)|$

$= 91$

20. If the system of equations

$2x - y + z = 4$

$5x + \lambda y + 3z = 12$

$100x - 47y + \mu z = 212,$

has infinitely many solutions, then $\mu - 2\lambda$ is equal to

(1) 56 (2) 59

(3) 55 (4) 57

Ans. **(4)**

Sol. $\Delta = 0 \Rightarrow \begin{vmatrix} 2 & -1 & 1 \\ 5 & \lambda & 3 \\ 100 & -47 & \mu \end{vmatrix} = 0$

$2(\lambda\mu + 141) + (5\mu - 300) - 235 - 100\lambda = 0$ ...(1)

$\Delta_3 = 0 \Rightarrow \begin{vmatrix} 2 & -1 & 4 \\ 5 & \lambda & 12 \\ 100 & -47 & 212 \end{vmatrix} = 0$

$6\lambda = -12 \Rightarrow \lambda = -2$

Put $\lambda = 2$ in (1)

$2(-2\mu + 141) + 5\mu - 300 - 235 + 200 = 0$

$\mu = 53$

$\therefore$ 57

SECTION-B

21. Let f be a differentiable function such that

$2(x+2)^2 f(x) - 3(x+2)^2 = 10\int_0^x (t+2) f(t)\,dt,$

$x \geq 0$. Then f(2) is equal to ______.

Ans. **(19)**

Sol. Differentiate both sides

$4(x+2) f(x) + 2(x+2)^2 f'(x) - 6(x+2) = 10(x+2) f(x)$

$2(x+2)^2 f'(x) - 6(x+2) f(x) = 6(x+2)$

$(x+2)\frac{dy}{dx} - 3y = 3$

$\int \frac{dy}{dx} = 3\int \frac{dx}{x+2}$

$\ln(y+1) = 3\ln(x+2) + C$

$(y+1) = C(x+2)^3$

$f(0) = \frac{3}{2}$

$f(2) = 19$

22. If for some α, β; $\alpha \le \beta$, $\alpha + \beta = 8$ and $\sec^2(\tan^{-1}\alpha) + \text{cosec}^2(\cot^{-1}\beta) = 36$, then $\alpha^2 + \beta$ is ______.

Ans. (14)

Sol. If$(\tan(\tan^{-1}(\alpha)) + 1\ (\cot(\cot^{-1}\beta))^2 = 36$

$\alpha^2 + \beta^2 = 34$

$\alpha\beta = 15$

$\alpha = 3, \beta = 5$

$\therefore\ \alpha^2 + \beta = 9 + 5 = 14$

23. The number of 3-digit numbers, that are divisible by 2 and 3, but not divisible by 4 and 9, is

Ans. (125)

Sol. No, of 3 digits = 999 – 99 = 900

No. of 3 digit numbers divisible by 2 & 3 i.e. by 6

$\frac{900}{6} = 150$

No. of 3 digit numbers divisible by 4 & 9 i.e. by 36

$\frac{900}{36} = 25$

$\therefore$ No of 3 digit numbers divisible by 2 & 3 but not by 4 & 9

150 – 25 = 125

24. Let be a 3 × 3 matrix such that $X^T AX = O$ for all nonzero 3 × 1 matrices $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

If $A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -5 \end{bmatrix}, A\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -8 \end{bmatrix}$, and

$\det(\text{adj}(2(A + I))) = 2^\alpha 3^\beta 5^\gamma$, $\alpha, \beta, \gamma, \in N$, then $\alpha^2 + \beta^2 + \gamma^2$ is

Ans. (44)

Sol. $X^TAX = 0$

$$(xyz)\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$(xyz)\begin{pmatrix} a_1x + a_2y + a_3z \\ b_1x + b_2y + b_3z \\ c_1x + c_2y + c_3z \end{pmatrix} = 0$$

$x(a_1x + a_2y + a_3z) + y(b_1x + b_2y + b_3z)$
$+ z(c_1x + c_2y + c_3z) = 0$

$a_1 = 0, b_2 = 0\ c_3 = 0$

$a_2 + b_1 = 0, a_3 + c_1 = 0, b_3 = c_2 = 0$

A = skew symm matrix

$$A = \begin{pmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{pmatrix}\ ; \quad A = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -5 \end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -5 \end{pmatrix}$$

$x + y = 1$

$-x + z = 4$

$y + z = 5$

$$\begin{pmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -8 \end{pmatrix}$$

$2x + y = 0 \qquad x = -1$

$-x + z = 4 \qquad y = 2$

$-y - 2z = -8 \qquad z = 3$

$$A = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ -2 & -3 & 0 \end{pmatrix}$$

$$2(A+I) = \begin{pmatrix} 2 & -2 & 4 \\ 2 & 2 & 6 \\ -2 & -6 & 2 \end{pmatrix}$$

$2(A+I) = 120 \quad \Rightarrow \det|\text{adi}(2(A+I))|$

$= 120^2 = 2^6.3^2.5^2$

$\alpha = 6, \beta = 2, \gamma = 2$

25. Let $S = \{p_1, p_2, \ldots\ldots, p_{10}\}$ be the set of first ten prime numbers. Let $A = S \cup P$, where P is the set of all possible products of distinct element of S. Then the number of all ordered pairs (x, y), $x \in S$, $y \in A$, such that x divides y , is ______.

Ans. (5120)

Sol. Let $\frac{y}{x} = \lambda$

$y = \lambda x$

$= 10 \times ({}^9C_0 + {}^9C_1 + {}^9C_2 + {}^9C_3 + \ldots. + {}^9C_9)$

$= 10 \times (2^9)$

10×512

5120

JEE–MAIN EXAMINATION – JANUARY 2025

(HELD ON FRIDAY 24th JANUARY 2025) **TIME : 3:00 PM TO 6:00 PM**

MATHEMATICS | TEST PAPER WITH SOLUTION

SECTION-A

1. The equation of the chord, of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$, whose mid-point is (3,1) is :

(1) $48x + 25y = 169$ (2) $4x + 122y = 134$

(3) $25x + 101y = 176$ (4) $5x + 16y = 31$

Ans. (1)

Sol. Equation of chord with given middle point

$T = S_1$

$\Rightarrow \frac{3x}{25}+\frac{y}{16}-1=\frac{9}{25}+\frac{1}{16}-1$

$48x + 25y = 144 + 25$

$48x + 25y = 169$ Ans.

2. The function $f : (-\infty, \infty) \to (-\infty, 1)$, defined by $f(x)=\frac{2^x-2^{-x}}{2^x+2^{-x}}$ is :

(1) One-one but not onto

(2) Onto but not one-one

(3) Both one-one and onto

(4) Neither one-one nor onto

Ans. (1)

Sol. $f(x) = \frac{2^{2x}-1}{2^{2x}+1}$

$=1-\frac{2}{2^{2x}+1}$

$f'(x) = \frac{2}{\left(2^{2x}+1\right)^2}.2.2^{2x}.\ell n2$ i.e always +ve

so f(x) is ↑ function

$\therefore f(-\infty) = -1$

$f(\infty) = 1$

$\therefore f(x) \in (-1, 1) \neq$ co-domain

so function is one-one but not onto

3. If $\alpha > \beta > \gamma > 0$, then the expression

$$\cot^{-1}\left\{\beta+\frac{(1+\beta^2)}{(\alpha-\beta)}\right\}+\cot^{-1}\left\{\gamma+\frac{(1+\gamma^2)}{(\beta-\gamma)}\right\}+\cot^{-1}\left\{\alpha+\frac{(1+\alpha^2)}{(\gamma-\alpha)}\right\}$$ is equal to:

(1) $\frac{\pi}{2}-(\alpha+\beta+\gamma)$ (2) 3π

(3) 0 (4) π

Ans. (4)

Sol. $\Rightarrow \cot^{-1}\left(\frac{\alpha\beta+1}{\alpha-\beta}\right)+\cot^{-1}\left(\frac{\beta\gamma+1}{\beta-\gamma}\right)+\cot^{-1}\left(\frac{\alpha\gamma+1}{\gamma-\alpha}\right)$

$\Rightarrow \tan^{-1}\left(\frac{\alpha-\beta}{1+\alpha\beta}\right)+\tan^{-1}\left(\frac{\beta-\gamma}{1+\beta\gamma}\right)+\pi+\tan^{-1}\left(\frac{\gamma-\alpha}{1+\gamma\alpha}\right)$

$\Rightarrow\left(\tan^{-1}\alpha-\tan^{-1}\beta\right)+\left(\tan^{-1}\beta-\tan^{-1}\gamma\right)+\left(\pi+\tan^{-1}\gamma-\tan^{-1}\alpha\right)$

$\Rightarrow \pi$

4. Let $f : (0, \infty) \to \mathbf{R}$ be a function which is differentiable at all points of its domain and satisfies the condition $x^2f'(x) = 2xf(x) + 3$, with $f(1) = 4$. Then $2f(2)$ is equal to:

(1) 29 (2) 19

(3) 39

(4) 23

Ans. (3)

Sol. $x^2f'(x) - 2x f(x) = 3$

$\left(\frac{x^2f'(x)-2xf(x)}{\left(x^2\right)^2}\right)=\frac{3}{\left(x^2\right)^2}$

$\Rightarrow \frac{d}{dx}\left(\frac{f(x)}{x^2}\right)=\frac{3}{x^4}$

Integrating both sides

$\frac{f(x)}{x^2}=-\frac{1}{x^3}+C$

$f(x) = -\frac{1}{x}+Cx^2$

put $x = 1$

$4 = -1 + C \Rightarrow C = 5$

$f(x) = -\frac{1}{x}+5x^2$

Now $2 \times f(2) = 2\times\left[-\frac{1}{2}+5\times2^2\right]$

$= 39$

5. Let

$$A=\left\{x\in(0,\pi)-\left\{\frac{\pi}{2}\right\}:\log_{(2/\pi)}|\sin x|+\log_{(2/\pi)}|\cos x|=2\right\}$$

and

$$B=\left\{x\ge 0:\sqrt{x}(\sqrt{x}-4)-3\left|\sqrt{x}-2\right|+6=0\right\}.$$ Then

$n(A\cup B)$ is equal to:

(1) 4 (2) 2

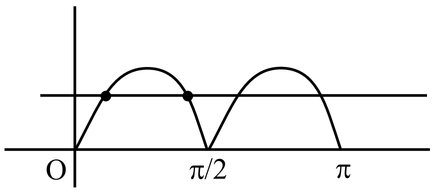
(3) 8 (4) 6

Ans. **(3)**

Sol. $A:\log_{2/\pi}|\sin x|+\log_{2/\pi}|\cos x|=2$

$\Rightarrow \log_{2/\pi}(|\sin x\,.\,\cos x|)=2$

$\Rightarrow |\sin 2x|=\dfrac{8}{\pi^2}$



Number of solution 4

B : let $\sqrt{x}=t<2$

Then $\sqrt{x}\left(\sqrt{x}-4\right)+3\left(\sqrt{x}-2\right)+6=0$

$\Rightarrow t^2-4t+3t-6+6=0$

$\Rightarrow t^2-t=0,\ t=0,\ t=1$

$x=0,\ x=1$

again let $\sqrt{x}=t>2$

then $t^2-4t-3t+6+6=0$

$\Rightarrow t^2-7t+12=0$

$\Rightarrow t=3,4$

$x=9,16$

Total number of solutions

$n(A\cup B)=4+4=8$

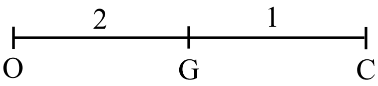
6. Let the position vectors of three vertices of a triangle be $4\vec{p}+\vec{q}-3\vec{r},\ -5\vec{p}+\vec{q}+2\vec{r}$ and $2\vec{p}-\vec{q}+2\vec{r}$. If the position vectors of the orthocenter and the circumcenter of the triangle are $\dfrac{\vec{p}+\vec{q}+\vec{r}}{4}$ and $\alpha\vec{p}+\beta\vec{q}+\gamma\vec{r}$ respectively, then $\alpha+2\beta+5\gamma$ is equal to:

(1) 3 (2) 1

(3) 6 (4) 4

Ans. **(1)**

Sol. We know that



O (orthocentre) $\dfrac{\vec{p}+\vec{q}+\vec{r}}{4}$

C (circum centre) $\alpha\vec{p}+\beta\vec{q}+\gamma\vec{r}$

C (centroid) $=\dfrac{\vec{p}+\vec{q}+\vec{r}}{3}$

by relation

$\Rightarrow 2\left(\alpha\vec{p}+\beta\vec{q}+\gamma\vec{r}\right)+\dfrac{\vec{p}+\vec{q}+\vec{r}}{4}=3\left(\dfrac{\vec{p}+\vec{q}+\vec{r}}{3}\right)$

$\Rightarrow 8\left(\alpha\vec{p}+\beta\vec{q}+\gamma\vec{r}\right)=3\left(\vec{p}+\vec{q}+\vec{r}\right)$

$\Rightarrow 8\alpha=3,\ 8\beta=3,\ 8\gamma=3$

$\alpha=\dfrac{3}{8},\ \beta=\dfrac{3}{8},\ \gamma=\dfrac{3}{8}$

$\therefore\ \alpha+2\beta+3\gamma$

$\dfrac{3}{8}+\dfrac{6}{8}+\dfrac{15}{8}=\dfrac{24}{8}=3$

7. Let [x] denote the gereatest integer function, and let m and n respectively be the numbers of the points, where the function $f(x)=[x]+|x-2|$, $-2<x<3$, is not continuous and not differentiable. Then m + n is equal to:

(1) 6 (2) 9

(3) 8 (4) 7

Ans. **(3)**

Sol. $f(x)=[x]+|x-2| \quad -2<x<3$

$$f(x)=\begin{cases}-x, & -2<x<-1\\ -x+1, & -1\le x<0\\ -x+2, & 0\le x<1\\ -x+3, & 1\le x<2\\ x, & 2\le x<3\end{cases}$$

So f(x) is not continuous at 4 points and not differentiable at 4 point

So $m+n=4+4=8$

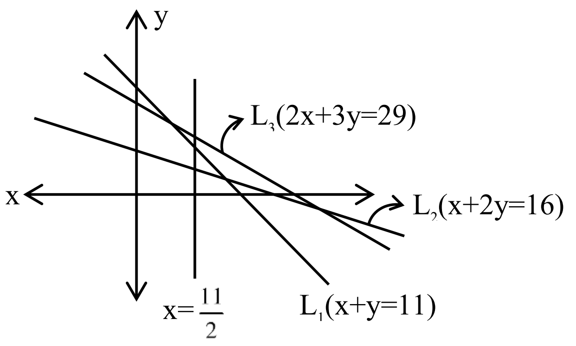
8. Let the points $\left(\frac{11}{2}, \alpha\right)$ lie on or inside the triangle with sides $x + y = 11$, $x + 2y = 16$ and $2x + 3y = 29$. Then the product of the smallest and the largest values of α is equal to :

(1) 22 (2) 44

(3) 33 (4) 55

Ans. **(3)**

Sol.



Point of intersection of $x = \frac{11}{2}$ with L_1 & L_3

gives, $\alpha_{min} = \frac{11}{2}$

and $\alpha_{max} = 6$

$\therefore \alpha_{min} \cdot \alpha_{max} = \frac{11}{2} \times 6 = 33$

9. In an arithmetic progression, if $S_{40} = 1030$ and $S_{12} = 57$, then $S_{30} - S_{10}$ is equal to:

(1) 510 (2) 515

(3) 525 (4) 505

Ans. **(2)**

Sol. Let a & d are first term and common diff of an AP.

$S_{40} = \frac{40}{2}[2a + 39d] = 1030$ ....(1)

$S_{12} = \frac{12}{2}[2a + 11d] = 57$ ....(2)

by (1) & (2)

$a = -\frac{7}{2}$ $\quad d = \frac{3}{2}$

$\therefore S_{30} - S_{10} = \frac{30}{2}[2a + 29d] - \frac{10}{2}[2a + 9d]$

$= 20a - 390d$

$= 515$

10. If $7 = 5 + \frac{1}{7}(5 + \alpha) + \frac{1}{7^2}(5 + 2\alpha) + \frac{1}{7^3}(5 + 3\alpha) + \ldots\ldots \infty$, then the value of α is:

(1) 1 (2) $\frac{6}{7}$

(3) 6 (4) $\frac{1}{7}$

Ans. **(3)**

Sol. Let $S = 5 + \frac{1}{7}(5 + \alpha) + \frac{1}{7^2}(5 + 2\alpha) + \ldots$

$\frac{1}{7}S = \frac{1}{7}(5) + \frac{1}{7^2}(5 + \alpha) + \ldots \infty$

$\frac{6}{7}(S) = 5 + \frac{1}{7}\alpha\left(\frac{1}{1 - \frac{1}{7}}\right)$

$6 = 5 + \frac{\alpha}{6} \Rightarrow \alpha = 6$

11. If the system of equations

$x + 2y - 3z = 2$

$2x + \lambda y + 5z = 5$

$14x + 3y + \mu z = 33$

has infinitely many solutions, then $\lambda + \mu$ is equal to:

(1) 13 (2) 10

(3) 11 (4) 12

Ans. **(4)**

Sol. $D = \begin{vmatrix} 1 & 2 & -3 \\ 2 & \lambda & 5 \\ 14 & 3 & \mu \end{vmatrix} = 0$, $\lambda\mu + 42\lambda - 4\mu + 107 = 0$

$D_1 = 2\lambda\mu + 99\lambda - 10\mu + 255$

$D_2 = 13 - \mu$

$D_3 = 5\lambda + 5$

$D_2 = 0 \Rightarrow \mu = 13$ & $D_3 = 0 \Rightarrow \lambda = -1$

check & verify for these values D & $D_2 = 0$

12. Let (2, 3) be the largest open interval in which the function $f(x) = 2\log_e(x - 2) - x^2 + ax + 1$ is strictly increasing and (b, c) be the largest open interval, in which the function $g(x) = (x - 1)^3(x + 2 - a)^2$ is strictly decreasing. Then $100(a + b - c)$ is equal to:

(1) 280 (2) 360

(3) 420 (4) 160

Ans. **(2)**

Sol. $f'(x)=\dfrac{2}{x-2}-2x+a\geq 0$

$f''(x)=\dfrac{-2}{(x-2)^2}-2<0$

$f'(x)\downarrow$

$f'(3)\geq 0$

$2-6+a\geq 0$

$a\geq 4$

$a_{min}=4$

$g(x)=(x-1)^3(x+2-a)^2$

$g(x)=(x-1)^3(x-2)^2$

$g'(x)=(x-1)^3\,2(x-2)+(x-2)^2\,3(x-1)^2$

$=(x-1)^2(x-2)(2x-2+3x-6)$

$=(x-1)^2(x-2)(5x-8)<0$

$x\in\left(\dfrac{8}{5},2\right)$

$100(a+b-c)=100\left(4+\dfrac{8}{5}-2\right)=360$

13. Suppose A and B are the coefficients of 30th and 12th terms respectively in the binomial expansion of $(1+x)^{2n-1}$. If 2A = 5B, then n is equal to:

(1) 22 (2) 21

(3) 20 (4) 19

Ans. **(2)**

Sol. $A={}^{2n-1}C_{29}$ $B={}^{2n-1}C_{11}$

$2\ {}^{2n-1}C_{29}=5\ {}^{2n-1}C_{11}$

$2\dfrac{(2n-1)!}{29!(2n-30)!}=5\dfrac{(2n-1)!}{(2n-12)!11!}$

$\dfrac{1}{29...12\cdot 5}=\dfrac{1}{(2n-12)(2n-13)...(2n-29)2}$

$\dfrac{1}{30\cdot 29...12}=\dfrac{1}{(2n-12)(2n-13)...(2n-29)12}$

$2n-12=30$

$n=21$

14. Let $\vec{a}=3\hat{i}-\hat{j}+2\hat{k},\ \vec{b}=\vec{a}\times(\hat{i}-2\hat{k})$ and $\vec{c}=\vec{b}\times\hat{k}$. Then the projection of $\vec{c}-2\hat{j}$ on $\vec{a}$ is:

(1) $3\sqrt{7}$

(2) $\sqrt{14}$

(3) $2\sqrt{14}$

(4) $2\sqrt{7}$

Ans. **(3)**

Sol. $\vec{b}=\vec{a}\times(\hat{i}-3\hat{k})$

$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & -1 & 2\\ 1 & 0 & -2\end{vmatrix}=2\hat{i}+8\hat{j}+\hat{k}$

$\vec{c}=\vec{b}\times\hat{k}=8\hat{i}-2\hat{j}$

$\vec{c}-2\hat{j}=8\hat{i}-4\hat{j}$

Projection of $(\hat{i}-2\hat{j})$ on $\vec{a}$

$(\vec{c}-2\hat{j}).\hat{a}=\dfrac{\langle 8,-4,0\rangle.\langle 3,-1,2\rangle}{\sqrt{14}}$

$=\dfrac{28}{\sqrt{14}}=2\sqrt{14}$

15. For some a, b, let

$$f(x)=\begin{vmatrix}a+\dfrac{\sin x}{x} & 1 & b\\ a & 1+\dfrac{\sin x}{x} & b\\ a & 1 & b+\dfrac{\sin x}{x}\end{vmatrix},\quad x\neq 0,$$

$\lim\limits_{x\to 0}f(x)=\lambda+\mu a+\nu b$. Then $(\lambda+\mu+\nu)^2$ is equal to:

(1) 25 (2) 9

(3) 36 (4) 16

Ans. **(4)**

Sol. $\lim\limits_{x\to 0}f(x)=\begin{vmatrix}a+1 & 1 & b\\ a & 1+1 & b\\ a & 1 & b+1\end{vmatrix}$

$=(a+1)(2(b+1)-b)+1(ab-a(b+1))+ba$

$=(a+1)(b+2)-a+ab$

$=b+a+2=\lambda+\mu a+\nu b$

$\lambda=2,\ \mu=1,\ \nu=1\ \Rightarrow(\lambda+\mu+\nu)^2=16$

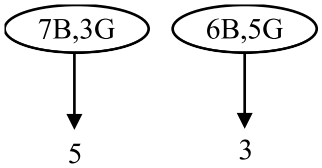
16. Group A consists of 7 boys and 3 girls, while group B consists of 6 boys and 5 girls. The number of ways, 4 boys and 4 girls can be invited for a picnic if 5 of them must be from group A and the remaining 3 from group B, is equal to:

(1) 8575 (2) 9100

(3) 8925 (4) 8750

Ans. (3)

Sol.



C-I (3G & 2B) & (1G & 2B)

C-II (2G & 3B) & (2G & 1B)

C-III (1G & 4B) & (3G & 0B)

Total = C-I + C-II + C-III

$= {}^{7}C_{2}\cdot{}^{3}C_{3}\cdot{}^{6}C_{2}\cdot{}^{5}C_{1} + {}^{7}C_{3}\cdot{}^{3}C_{2}\cdot{}^{6}C_{1}\,{}^{5}C_{2} + {}^{7}C_{4}\cdot{}^{3}C_{1}\cdot{}^{6}C_{0}\cdot{}^{5}C_{3}$

= 8925

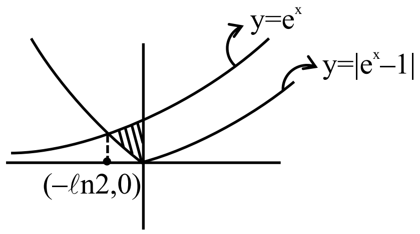
17. The area of the region enclosed by the curves $y = e^x$, $y = |e^x - 1|$ and y-axis is:

(1) $1 + \log_e 2$ (2) $\log_e 2$

(3) $2\log_e 2 - 1$ (4) $1 - \log_e 2$

Ans. (4)

Sol.



For Area $\int_{-\ell n2}^{0}\left[e^x - \left(1 - e^x\right)\right]dx$

$\int_{-\ell n2}^{0}\left(2e^x - 1\right)dx = \left[2e^x - x\right]_{-\ell n2}^{0}$

$= (2 - (1 + \ell n2))$

$= 1 - \ell n2$

18. The number of real solution(s) of the equation $x^2 + 3x + 2 = \min\{|x - 3|, |x + 2|\}$ is :

(1) 2

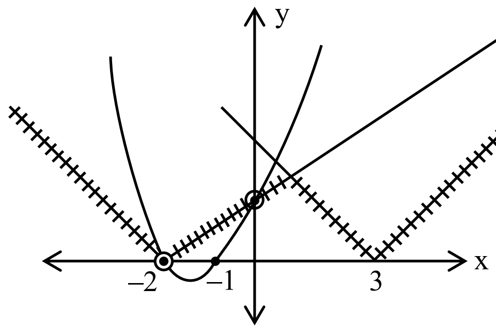
(2) 0

(3) 3

(4) 1

Ans. (1)

Sol.



Only 2 solutions.

19. Let $A = [a_{ij}]$ be a square matrix of order 2 with entries either 0 or 1. Let E be the event that A is an invertible matrix. Then the probability P(E) is :

(1) $\frac{5}{8}$ (2) $\frac{3}{16}$

(3) $\frac{1}{8}$ (4) $\frac{3}{8}$

Ans. (4)

Sol. C-I $\begin{vmatrix}1 & 1\\0 & 1\end{vmatrix} \rightarrow$ 4 ways

C-II $\begin{vmatrix}1 & 0\\0 & 1\end{vmatrix}$ & $\begin{vmatrix}0 & 1\\1 & 0\end{vmatrix} \rightarrow$ 2 ways

$P = \frac{\text{favourable}}{\text{total}} = \frac{6}{16} = \frac{3}{8}$

20. If the equation of the parabola with vertex $V\left(\frac{3}{2}, 3\right)$ and the directrix $x + 2y = 0$ is $\alpha x^2 + \beta y^2 - \gamma xy - 30x - 60y + 225 = 0$, then $\alpha + \beta + \gamma$ is equal to:

(1) 6

(2) 8

(3) 7

(4) 9

Ans. (4)

Sol. Equation of axis $y - 3 = 2\left(x - \frac{3}{2}\right)$

$y - 2x = 0$

foot of directrix

$y - 2x = 0$

& $\Rightarrow (0, 0)$

$2y + x = 0$

Focus = (3, 6)

$PS^2 = PM^2$

$(x-3)^2 + (y-6)^2 = \left(\frac{x+2y}{\sqrt{5}}\right)^2$

$4x^2 + y^2 - 4xy - 30x - 60y + 225 = 0$

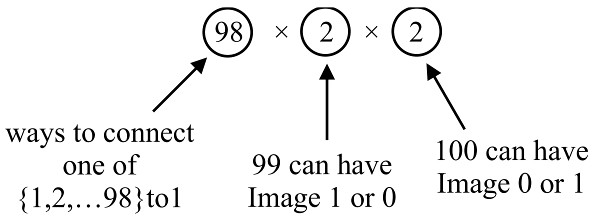
$\Rightarrow \alpha = 4, \beta = 1, \gamma = 4 \Rightarrow \alpha + \beta + \gamma = 9$

SECTION-B

21. Number of functions $f : \{1, 2, \ldots, 100\} \to \{0, 1\}$, that assign 1 to exactly one of the positive integers less than or equal to 98, is equal to ______.

Ans. (392)

Sol.

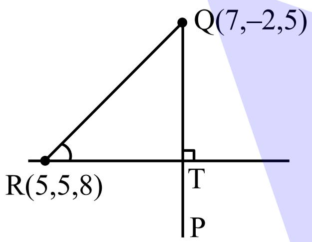


392 Ans.

22. Let P be the image of the point Q(7, –2, 5) in the line $L : \frac{x-1}{2} = \frac{y+1}{3} = \frac{z}{4}$ and R(5, p, q) be a point on L. Then the square of the area of ΔPQR is ____

Ans. (957)

Sol.



Let R(2λ + 1, 3λ – 1, 4λ)

2λ + 1 = 5

λ = 2

R(5, 5, 8)

let T(2λ + 1, 3λ – 1, 4λ)

$\overrightarrow{QT} = (2\lambda-6)\hat{i} + (3\lambda+1)\hat{j} + (4\lambda-5)\hat{k}$

$\vec{b} = 2\hat{i} + 3\hat{j} + 4\hat{k}$

$\overrightarrow{QT} \cdot \vec{b} = 0$

$4\lambda - 12 + 9\lambda + 3 + 16\lambda - 20 = 0$

λ = 1

T(3, 2, 4)

$QT = \sqrt{33}$ $\quad RT = \sqrt{29}$

$(\text{area of } \Delta PQR)^2 = \left(\frac{1}{2}\sqrt{29}.2\sqrt{33}\right)^2$

= 957

23. Let y = y(x) be the solution of the differential equation $2\cos x\frac{dy}{dx} = \sin 2x - 4y \sin x$, $x \in \left(0, \frac{\pi}{2}\right)$. If $y\left(\frac{\pi}{3}\right) = 0$, then $y'\left(\frac{\pi}{4}\right) + y\left(\frac{\pi}{4}\right)$ is equal to ____.

Ans. (1)

Sol. $\frac{dy}{dx} + 2y\tan x = \sin x$

$\text{I.F.} = e^{2\int \tan x\, dx} = \sec^2 x$

$y\sec^2 x = \int \frac{\sin x}{\cos^2 x}dx$

$= \int \tan x \sec x\, dx$

= secx + C

C = –2

$y = \cos x - 2\cos^2 x$

$y\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} - 1$

y′ = –sinx + 4cosx sinx

$y'\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} + 2$

$y'\left(\frac{\pi}{4}\right) + y\left(\frac{\pi}{4}\right) = 1$

24. Let $H_1 : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $H_2 : -\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$ be two hyperbolas having length of latus rectums $15\sqrt{2}$ and $12\sqrt{5}$ respectively. Let their eccentricities be $e_1 = \sqrt{\frac{5}{2}}$ and e_2 respectively. If the product of the lengths of their transverse axes is $100\sqrt{10}$, then $25e_2^2$ is equal to ______

Ans. (55)

Sol. $\dfrac{2b^2}{a}=15\sqrt{2}$

$1+\dfrac{b^2}{a^2}=\dfrac{5}{2}$

$a=5\sqrt{2}$

$b=5\sqrt{3}$

$\dfrac{2A^2}{B}=12\sqrt{5}$

$2a.2B = 100\sqrt{10}$

$2.5\sqrt{2}.2B=100\sqrt{10}$

$B = 5\sqrt{5}$

$A = 5\sqrt{6}$

$e_2^2=1+\dfrac{A^2}{B^2}$

$=1+\dfrac{150}{125}$

$e_2^2=1+\dfrac{30}{25}$

$25e_2^2=55$

25. If $\int \dfrac{2x^2+5x+9}{\sqrt{x^2+x+1}}dx = x\sqrt{x^2+x+1}+\alpha\sqrt{x^2+x+1}+$ $\beta\log_e\left|x+\dfrac{1}{2}+\sqrt{x^2+x+1}\right|+C$, where C is the constant of integration, then $\alpha + 2\beta$ is equal to ___

Ans. **(16)**

Sol. $2x^2 + 5x + 9 = A(x^2 + x + 1) + B(2x + 1) + C$

$A = 2 \qquad B = \dfrac{3}{2} \qquad C = \dfrac{11}{2}$

$2\int\sqrt{x^2+x+1}\,dx+\dfrac{3}{2}\int\dfrac{2x+1}{\sqrt{x^2+x+1}}dx+\dfrac{11}{2}\int\dfrac{dx}{\sqrt{x^2+x+1}}$

$2\int\sqrt{\left(x+\dfrac{1}{2}\right)^2+\left(\dfrac{\sqrt{3}}{2}\right)^2}dx+3\sqrt{x^2+x+1}+\dfrac{11}{2}\int\dfrac{dx}{\sqrt{\left(x+\dfrac{1}{2}\right)^2+\left(\dfrac{\sqrt{3}}{2}\right)^2}}$

$2\left(\dfrac{x+\frac{1}{2}}{2}\sqrt{x^2+x+1}+\dfrac{3}{8}\ell n\left(x+\dfrac{1}{2}+\sqrt{x^2+x+1}\right)\right)+3\sqrt{x^2+x+1}$

$+\dfrac{11}{2}\ell n\left(x+\dfrac{1}{2}+\sqrt{x^2+x+1}\right)+C$

$\alpha=\dfrac{7}{2} \qquad \beta=\dfrac{25}{4}$

$\alpha + 2\beta = 16$

JEE–MAIN EXAMINATION – JANUARY 2025

(HELD ON TUESDAY 28th JANUARY 2025) **TIME : 9:00 AM TO 12:00 NOON**

MATHEMATICS | TEST PAPER WITH SOLUTION

SECTION-A

1. The number of different 5 digit numbers greater than 50000 that can be formed using the digits 0, 1, 2, 3, 4, 5, 6, 7, such that the sum of their first and last digits should not be more than 8, is

(1) 4608 (2) 5720

(3) 5719 (4) 4607

Ans. (4)

Sol.

Case I	5 _ _ _	0
Case II	5 _ _ _	1
	5	2
	5	3
	6	0
	6	1
	6	2
	7	0
	7 _ _ _	1

$9\times(8\times8\times8) = 4608$ but 50000 is not included, so total numbers $4608 - 1 = 4607$

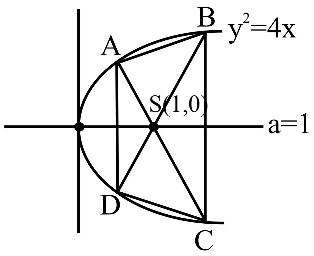
2. Let ABCD be a trapezium whose vertices lie on the parabola $y^2 = 4x$. Let the sides AD and BC of the trapezium be parallel to y-axis. If the diagonal AC is of length $\frac{25}{4}$ and it passes through the point (1, 0), then the area of ABCD is :

(1) $\frac{75}{4}$ (2) $\frac{25}{2}$

(3) $\frac{125}{8}$ (4) $\frac{75}{8}$

Ans. (1)

Sol.



$A(at_1^2, 2at_1)$ & $C\left(\frac{a}{t_1^2}, -\frac{2a}{t_1}\right)$

Length $AC = a\left(t_1 + \frac{1}{t_1}\right)^2 = \frac{25}{4}$, $t_1 + \frac{1}{t_1} = \pm\frac{5}{2}$

$\Rightarrow t_1 = 2$ or $\frac{1}{2}$, $A\left(\frac{1}{2}, 1\right)$, $D\left(\frac{1}{4}, -1\right)$, B(4, 4), C(4, –4)

So, area of trapezium = $\frac{1}{2}(8+2)\left(4 - \frac{1}{4}\right) = \frac{75}{4}$

3. Two number k_1 and k_2 are randomly chosen from the set of natural numbers. Then, the probability that the value of $i^{k_1} + i^{k_2}$, $(i = \sqrt{-1})$ is non-zero, equals

(1) $\frac{1}{2}$ (2) $\frac{1}{4}$

(3) $\frac{3}{4}$ (4) $\frac{2}{3}$

Ans. (3)

Sol. $i^{k_1} + i^{k_2} \neq 0$ $\quad i^{k_1} \to 4$ option for i, –1, –i, 1

Total cases $\Rightarrow 4\times4 = 16$

Unfovourble cases $\Rightarrow i^{k_1} + i^{k_2} = 0$

$$\begin{Bmatrix} 1, -1 \\ -1, 1 \\ i, -i \\ -i, i \end{Bmatrix}$$

4 Cases $\Rightarrow$ Probability = $\frac{16-4}{16} = \frac{3}{4}$

4. If $f(x) = \frac{2^x}{2^x + \sqrt{2}}, x \in R$, then $\sum_{k=1}^{81} f\left(\frac{k}{82}\right)$ is equal to :

(1) 41 (2) $\frac{81}{2}$

(3) 82 (4) $81\sqrt{2}$

Ans. (2)

Sol. $f(x)=\frac{2^x}{2^x+\sqrt{2}}$

$f(x)+f(1-x)=\frac{2^x}{2^x+\sqrt{2}}+\frac{2^{1-x}}{2^{1-x}+\sqrt{2}}$

$=\frac{2^x}{2^x+\sqrt{2}}+\frac{2}{2+\sqrt{2}\,2^x}=\frac{2^x+\sqrt{2}}{2^x+\sqrt{2}}=1$

Now, $\sum_{k=1}^{81}f\left(\frac{k}{82}\right)=f\left(\frac{1}{82}\right)+f\left(\frac{2}{82}\right)+.......+f\left(\frac{81}{82}\right)$

$=f\left(\frac{1}{82}\right)+f\left(\frac{1}{82}\right)+.......+f\left(1-\frac{2}{82}\right)+f\left(1-\frac{1}{82}\right)$

$\left[f\left(\frac{1}{82}\right)+f\left(1-\frac{1}{82}\right)\right]+\left[f\left(\frac{2}{82}\right)+f\left(1-\frac{2}{82}\right)\right]+.....40\text{ cases}+f\left(\frac{41}{82}\right)$

$(1+1+.....+1)\,40\text{ times}+\frac{2^{1/2}}{2^{1/2}+2^{1/2}}$

$40+\frac{1}{2}=\frac{81}{2}$

5. Let f : R → R be a function defined by $f(x)=(2+3a)x^2+\left(\frac{a+2}{a-1}\right)x+b,\ a\neq 1$. If $f(x+y)=f(x)+f(y)+1-\frac{2}{7}xy$, then the value of $28\sum_{i=1}^{5}|f(i)|$ is:

(1) 715 (2) 735

(3) 545 (4) 675

Ans. **(4)**

Sol. $f(x)=(3a+2)x^2+\left(\frac{a+2}{a-1}\right)x+b$

$f\left(x+\frac{1}{2}\right)=f(x)+f(y)+1-\frac{2}{7}xy$ ......(1)

In (1) Put $x=y=0\Rightarrow f(0)=2f(0)+1\Rightarrow f(0)=-1$

So, $f(0)=0+0+b=-1\Rightarrow b=-1$

In (1) Put $y=-x\Rightarrow f(0)=f(x)+f(-x)+1+\frac{2}{7}x^2$

$-1=2(3a+2)x^2+2b+1+\frac{2}{7}x^2$

$-1=\left(2(3a+2)+\frac{2}{7}\right)x^2+1-2$

$\Rightarrow 6a+4+\frac{2}{7}=0$

$a=-\frac{5}{7}$

So $f(x)=-\frac{1}{7}x^2-\frac{3}{4}x-1$

$\Rightarrow |f(x)|=\frac{1}{28}|4x^2+21x+28|$

Now, $28\sum_{i=1}^{5}|f(6)|=28(|f(1)|+f(2)+...+f(5))$

$28.\frac{1}{28}.675=675$

6. Let A(x, y, z) be a point in xy-plane, which is equidistant from three points (0, 3, 2), (2, 0, 3) and (0, 0, 1).

Let B = (1, 4, –1) and C = (2, 0, –2). Then among the statements

(S1) : ΔABC is an isosceles right angled triangle and

(S2) : the area of ΔABC is $\frac{9\sqrt{2}}{2}$.

(1) both are true (2) only (S1) is true

(3) only (S2) is true (4) both are false

Ans. **(2)**

Sol. A(x,y,z) Let P (0,3,2), Q(2,0,3), R(0,0,1)

AP = AQ = AR

$x^2+(y-3)^2+(z-2)^2=(x-2)^2+y^2+(z-3)^2=x^2+y^2+(z-1)^2$

In xy plane z = 0

So, $x^2-4x+4+y^2+9=x^2+y^2+1$

x = 3

$9+y^2-6y+9+4=x^2+y^2+1$

So, A(3,2,0) also B(1,4,–1) & C(2,0,–2)

Now $AB=\sqrt{4+4+1}=3$

$AC=\sqrt{1+4+4}=3$

$BC=\sqrt{1+16+1}=\sqrt{18}$

AB = AC

isosceles Δ & $AB^2 + AC^2 = BC^2$

right angle Δ

Area of ΔABC = $\frac{1}{2} \times$base.height

$\frac{1}{2} \times 3 \times 3 = \frac{9}{2}$

So only S_1 is true

7. The relation R={(x, y) : x, y ∈ z and x + y is even} is :

(1) reflexive and transitive but not symmetric

(2) reflexive and symmetric but not transitive

(3) an equivalence relation

(4) symmetric and transitive but not reflexive

Ans. **(3)**

Sol. R = {(x,y), x + y is even x, y ∈ z}

reflexive x + x = 2x even

symmetric of x + y is even, then (y + x) is also even

transitive of x + y is even & y + z is even then x + z is also even

So, relation is an equivalence relation.

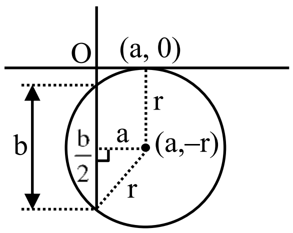
8. Let the equation of the circle, which touches x-axis at the point (a, 0), a > 0 and cuts off an intercept of length b on y-axis be $x^2 + y^2 - \alpha x + \beta y + \gamma = 0$. If the circle lies below x-axis, then the ordered pair $(2a, b^2)$ is equal to :

(1) $(\alpha, \beta^2 + 4\gamma)$ (2) $(\gamma, \beta^2 - 4\alpha)$

(3) $(\gamma, \beta^2 + 4\alpha)$ (4) $(\alpha, \beta^2 - 4\gamma)$

Ans. **(4)**

Sol.



By pytogorus $r^2 = a^2 + \frac{b^2}{4} = P^2$

$r = \sqrt{\frac{4a^2 + b^2}{4}}$

Equation of circle is $(x - \alpha)^2 + (y - \beta)^2 = r^2$

$x^2 + y^2 - 2ax - 2py + \alpha^2 + p^2 - r^2 = 0$

comparision $x^2 + y^2 - \alpha x + \beta y + r = 0$

$-\alpha = -2a,\ \beta = -2p,\ r = a^2$

$\Rightarrow 2a = \alpha,\ \ 4a^2 + b^2 = 4p^2$

$\alpha^2 + b^2 = 4p^2$

$\alpha^2 + b^2 = \beta^2$

So, $(2a, b^2) = (\alpha, \beta^2 - 4r)$

9. Let $<a_n>$ be a sequence such that $a_0 = 0$, $a_1 = \frac{1}{2}$ and $2a_{n+2} = 5a_{n+1} - 3a_n$, n = 0, 1, 2, 3, …… Then $\sum_{k=1}^{100} a_k$ is equal to :

(1) $3a_{99} - 100$ (2) $3a_{100} - 100$

(3) $3a_{100} + 100$ (4) $3a_{99} + 100$

Ans. **(2)**

Sol. $a_0 = 0,\ a_1 = \frac{1}{2}$

$2a_{n+2} = 5a_{n+1} - 3a_n$

$2x^2 - 5x + 3 = 0 \Rightarrow x = 1, 3/2$

$\therefore\ a_n = A1^n + B\left(\frac{3}{2}\right)^n$

$$\left.\begin{array}{ll} n=0 & 0 = A + B \\ n = 1 & \frac{1}{2} = A + \frac{3}{2}B \end{array}\right] \begin{array}{l} A = -1 \\ B = 1 \end{array}$$

$\Rightarrow a_n = -1 + \left(\frac{3}{2}\right)^n$

$\sum_{k=1}^{100} a_k = \sum_{k=1}^{100} (-1) + \left(\frac{3}{2}\right)^k$

$= -100 + \frac{\left(\frac{3}{2}\right)\left(\left(\frac{3}{2}\right)^{100} - 1\right)}{\frac{3}{2} - 1}$

$= -100 + 3\left(\left(\frac{3}{2}\right)^{100} - 1\right)$

$= 3.(a_{100}) - 100$

10. $\cos\left(\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{33}{65}\right)$ is equal to :

(1) 1 (2) 0

(3) $\frac{33}{65}$ (4) $\frac{32}{65}$

Ans. (2)

Sol. $\cos\left(\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{33}{65}\right)$

$\cos\left(\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{5}{12}+\tan^{-1}\frac{33}{56}\right)$

$\cos\left(\tan^{-1}\left(\frac{\frac{3}{4}+\frac{5}{12}}{1+\frac{3}{4}\cdot\frac{5}{12}}\right)+\tan^{-1}\frac{33}{56}\right)$

$\cos\left(\tan^{-1}\frac{56}{33}+\cot^{-1}\frac{56}{33}\right)$

$\cos\left(\frac{\pi}{2}\right)=0$

11. Let T_r be the r^{th} term of an A.P. If for some m, $T_m = \frac{1}{25}$, $T_{25} = \frac{1}{20}$ and $20\sum_{r=1}^{25} T_r = 13$, then $5m\sum_{r=m}^{2m} T_r$ is equal to :

(1) 112 (2) 126

(3) 98 (4) 142

Ans. (2)

Sol. $T_m = \frac{1}{25}$, $T_{25} = \frac{1}{20}$, $20\sum_{r=1}^{25} T_r = 13$

$T_m = a + (m-1)d = \frac{1}{25}$ ......(1)

$T_{25} = a + 24d = \frac{1}{20}$

$20.\frac{25}{2}\left[a+\frac{1}{20}\right]=13 \Rightarrow a=\frac{1}{500}$

also, $20S_{25} = 20.\frac{25}{2}[2a+24d]=13 \Rightarrow d=\frac{1}{500}$

from (1) $\frac{1}{500}+\frac{m-1}{500}=\frac{1}{25} \Rightarrow m=20$

Now,

$5m\sum_{r=m}^{2m} T_r = 100\sum_{r=20}^{40} T_r = 126$

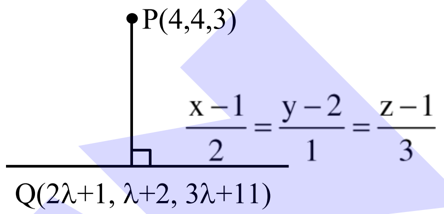
12. If the image of the point (4, 4, 3) in the line $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-1}{3}$ is (α, β, γ), then $\alpha+\beta+\gamma$ is equal to

(1) 9 (2) 12

(3) 8 (4) 7

Ans. (1)

Sol.



$\overrightarrow{P\alpha} \perp \left(2\hat{i}+\hat{j}+3\hat{k}\right)$

$\Rightarrow 2(2\lambda-3)+1(\lambda-2)+3(3\lambda-2)=0$

$\Rightarrow 14\lambda - 14 = 0, \lambda = 1$

So, Q(3,3,4)

Let image in $R(\alpha,\beta,\gamma)$

$\frac{\alpha+\gamma}{2}=3, \frac{\beta+\gamma}{2}=3, \frac{\gamma+3}{2}=4$

$(\alpha, \beta, \gamma) = (2, 2, 5)$

$\Rightarrow \alpha+\beta+\gamma = 9$

13. If $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{96x^2\cos^2 x}{(1+e^x)}dx = \pi(\alpha\pi^2+\beta)$, $\alpha, \beta \in Z$, then $(\alpha+\beta)^2$ equals :

(1) 144 (2) 196

(3) 100 (4) 64

Ans. (3)

Sol. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{96x^2\cos^2 x}{(1+e^x)}dx$ (Apply King Property)

$\int_0^{\frac{\pi}{2}} 96x^2\cos^2 x = 48\int_0^{\frac{\pi}{2}} x^2(1+\cos 2x)dx$

$48\left[\left(\frac{x^3}{3}\right)_0^{\pi/2}+\int_0^{\frac{\pi}{2}} \underset{I}{x^2}\,\underset{II}{\cos 2x}\,dx\right]$

$\Rightarrow$ On solving $\pi(2\pi^2-12)$

$\Rightarrow \alpha = 2, \beta = -12$

$\Rightarrow (\alpha+\beta)^2 = 100$

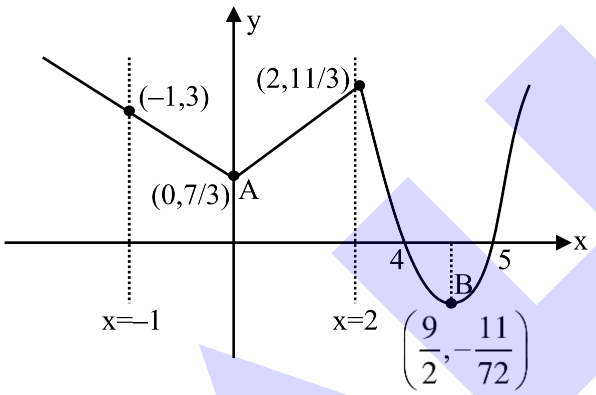
14. The sum of all local minimum values of the function $f(x)=\begin{cases}1-2x, & x<-1\\ \frac{1}{3}(7+2|x|), & -1\le x\le 2\\ \frac{11}{18}(x-4)(x-5), & x>2\end{cases}$

(1) $\frac{171}{72}$ (2) $\frac{131}{72}$

(3) $\frac{157}{72}$ (4) $\frac{167}{72}$

Ans. (3)

Sol. $f(x)=\begin{cases}1-2x, & x<-1\\ \frac{1}{3}(7-2x), & -1\le x\le 2\\ \frac{1}{3}(7+2x) & 0\le x<2\\ \frac{11}{18}(x-4)(x-5), & x>2\end{cases}$



∴ Local minimum values at A & B

$\frac{7}{3}-\frac{11}{72}$

$\Rightarrow \frac{168-11}{72}\Rightarrow\frac{157}{72}$

15. The sum, of the squares of all the roots of the equation $x^2+|2x-3|-4=0$, is :

(1) $3(3-\sqrt{2})$ (2) $6(3-\sqrt{2})$

(3) $6(2-\sqrt{2})$ (4) $3(2-\sqrt{2})$

Ans. (3)

Sol. $x^2+|2x-3|-4=0$

Case I : $x\ge\frac{3}{2}$

$x^2+2x-3-4=0$

$x^2+2x-7=0$

$x=2\sqrt{2}-1$

Case II : $x<\frac{3}{2}$

$x^2+3-2x-4=0$

$x^2-2x-1=0$

$x=1-\sqrt{2}$

Sum of squares $=\left(2\sqrt{2}-1\right)^2+\left(1-\sqrt{2}\right)^2$

$=8-4\sqrt{2}+1+1-2\sqrt{2}+2$

$=6\left(2-\sqrt{2}\right)$ ∴ (3)

16. Let for some function $y=f(x)$, $\int_0^x t\,f(t)dt=x^2f(x)$, $x>0$ and $f(2)=3$. Then $f(6)$ is equal to :

(1) 1 (2) 2

(3) 6 (4) 3

Ans. (1)

Sol. $\int_0^x tf(t)dt=x^2+(x), x>0$

Diff. both side w.r. to x

$xf(x)=x^2f'(x)+2xf(x)$

$-xf(x)=x^2f'(x)$

$\int\frac{f'(x)}{f(x)}dx=\int\frac{-1}{2}dx$

$\log f(x)=-\log x+\log c$

$f(x)=\frac{c}{x}$

$f(2)=3\Rightarrow 3=\frac{c}{2}\Rightarrow c=6$

$f(x)=\frac{6}{x}$

$f(6)=1$ ∴ (1)

17. Let ${}^nC_{r-1}=28$, ${}^nC_r=56$ and ${}^nC_{r+1}=70$. Let $A(4\cos t, 4\sin t)$, $B(2\sin t, -2\cos t)$ and $C(3r-n, r^2-n-1)$ be the vertices of a triangle ABC, where t is a parameter. If $(3x-1)^2+(3y)^2=\alpha$, is the locus of the centroid of triangle ABC, then α equals :

(1) 20 (2) 8

(3) 6 (4) 18

Ans. (1)

Sol. ${}^nC_{r-1} = 28$, ${}^nC_r = 56$

$$\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{28}{56}$$

$$\frac{\dfrac{n!}{(r-1)!(n-r+1)!}}{\dfrac{n!}{r!(n-r)!}} = \frac{1}{2}$$

$$\frac{r}{(n-r+1)} = \frac{1}{2}$$

$3r = n+1$ —(i)

$$\frac{{}^nC_r}{{}^nC_{r+1}} = \frac{56}{70}$$

$$\frac{(r+1)}{(n-r)} = \frac{56}{70} \Rightarrow 9r = 4n-5 \quad \text{—(ii)}$$

By (i) & (ii)

$(r = 3)$, $(n = 8)$

A $(4\cos t, 4\sin t)$ B$(2\sin t, -2\cos t)$ C$(3r-n, r^2-n-1)$

A $(4\cos t, 4\sin t)$ B$(2\sin t, -2\cos t)$ C$(1, 0)$

$(3x-1)^2 + (3y)^2 = (4\cos t + 2\sin t)^2 + (4\sin t - \cos t)^2$

$(3x-1)^2 + (3y)^2 = 20 \quad \therefore (1)$

18. Let O be the origin, the point A be $z_1 = \sqrt{3} + 2\sqrt{2}i$, the point B$(z_2)$ be such that $\sqrt{3}|z_2| = |z_1|$ and $\arg(z_2) = \arg(z_1) + \frac{\pi}{6}$. Then

(1) area of triangle ABO is $\frac{11}{\sqrt{3}}$

(2) ABO is a scalene triangle

(3) area of triangle ABO is $\frac{11}{4}$

(4) ABO is an obtuse angled isosceles triangle

Ans. **(4)**

Sol. $z_1 = \sqrt{3} + 2\sqrt{2}i$ & $\frac{|z_2|}{|z_1|} = \frac{1}{\sqrt{3}}$

given $\arg\left(\frac{z_2}{z_1}\right) = \frac{\pi}{6}$

$$z_2 = \frac{|z_2|}{|z_1|} \cdot z_1\, e^{i\left(\frac{\pi}{6}\right)}$$

$$z_2 = \frac{1}{\sqrt{3}} \cdot \frac{\left(\sqrt{3}+2\sqrt{2}i\right)\left(\sqrt{3}+i\right)}{2}$$

$$z_2 = \frac{\left(3-2\sqrt{2}\right)+i\left(2\sqrt{6}+\sqrt{3}\right)}{2\sqrt{3}}$$

Now,

$$z_1 - z_2 = \frac{\left(3+2\sqrt{2}\right)+i\left(2\sqrt{6}-\sqrt{3}\right)}{2\sqrt{3}}$$

$|z_1-z_2| = |z_2| \Rightarrow \Delta ABO$ is isosceles with angles $\frac{\pi}{6}, \frac{\pi}{6}$ & $\frac{2\pi}{3}$

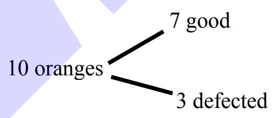
19. Three defective oranges are accidently mixed with seven good ones and on looking at them, it is not possible to differentiate between them. Two oranges are drawn at random from the lot. If x denote the number of defective oranges, then the variance of x is :

(1) 28/75 (2) 14/25

(3) 26/75 (4) 18/25

Ans. **(1)**

Sol.



Probability distribution

x_i	p_i
$x = 0$	$\frac{7C_2}{10C_2} = \frac{42}{90}$
$x = 1$	$\frac{7C_1 \times 3C_1}{10C_2} = \frac{42}{90}$
$x = 2$	$\frac{3C_2}{10C_2} = \frac{6}{90}$

Now,

$$\mu = \sum x_i p_i = \frac{42}{90} + \frac{12}{90} = \frac{54}{90}$$

$$\sigma^2 = \sum p_i x_i^2 - \mu^2 = \frac{42}{90} + \frac{24}{90} - \left(\frac{54}{90}\right)^2$$

$$\Rightarrow \frac{66}{90} - \left(\frac{54}{90}\right)^2$$

$$\sigma^2 \Rightarrow \frac{28}{75} \quad \therefore (1)$$

20. The area (in sq. units) of the region

$\{(x, y): 0 \le y \le 2|x| + 1, 0 \le y \le x^2 + 1, |x| \le 3\}$

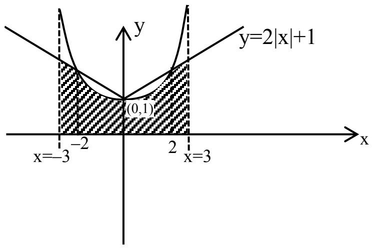
is

(1) $\frac{80}{3}$ (2) $\frac{64}{3}$

(3) $\frac{17}{3}$ (4) $\frac{32}{3}$

Ans. (2)

Sol.



$$\text{Area} = 2\left[\int_0^2 (x^2+1)dx + \int_2^3 (2x+1)dx\right]$$

$\Rightarrow \frac{64}{3}$ $\therefore$ (2)

SECTION-B

21. Let M denote the set of all real matrices of order 3×3 and let $S = \{-3, -2, -1, 1, 2\}$. Let

$S_1 = \{A = [a_{ij}] \in M : A = A^T \text{ and } a_{ij} \in S, \forall i, j\}$

$S_2 = \{A = [a_{ij}] \in M : A = -A^T \text{ and } a_{ij} \in S, \forall i, j\}$

$S_3 = \{A = [a_{ij}] \in M : a_{11} + a_{22} + a_{33} = 0 \text{ and } a_{ij} \in S, \forall i,j\}$

If $n(S_1 \cup S_2 \cup S_3) = 125\alpha$, then α equals.

Ans. (1613)

Sol. $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

No. of elements in $S_1 : A = A^T \Rightarrow 5^3 \times 5^3$

No. of elements in $A = -A^T \Rightarrow 0$

since no. zero in 5

No. of elements in $S_3 \Rightarrow$

$$\left.\begin{array}{l} a_{11} + a_{22} + a_{33} = 0 \Rightarrow (1,2,-3) \Rightarrow 31 \\ \text{or} \\ (1,1,-2) \Rightarrow 3 \\ \text{or} \\ (-1,-1,2) \Rightarrow 3 \end{array}\right\} \Rightarrow 12 \times 5^6$$

$n(S_1 \cap S_3) = 12 \times 5^3$

$n(S_1 \cup S_2 \cup S_3) = 5^6(1+12) - 12 \times 5^3$

$\Rightarrow 5^3 \times [13 \times 5^3 - 12] = 125\alpha$

$\alpha = 1613$

22. If $\alpha = 1 + \sum_{r=1}^{6} (-3)^{r-1}\ {}^{12}C_{2r-1}$' then the distance of the point $(12, \sqrt{3})$ form the line $\alpha x - \sqrt{3}y + 1 = 0$ is ....

Ans. (5)

Sol. $\alpha = 1 + \sum_{r=1}^{6} (-1)^{r-1}\ {}^{12}C_{2r-1} 3^{r-1}$

$\alpha = 1 + \sum_{r=1}^{6} {}^{12}C_{2r-1} \frac{(\sqrt{3}i)^{2t-1}}{\sqrt{3}i}$ $\quad i = \text{iota}, \text{let } \sqrt{3}\,i = x$

$\alpha = 1 + \frac{1}{\sqrt{3}i}\left({}^{12}C_1 x + {}^{12}C_3 x^3 + \ldots {}^{12}C_{11} x^{11}\right)$

$= 1 + \frac{1}{\sqrt{3}i}\left(\frac{(1+\sqrt{3}i)^{12} - (1-\sqrt{3}i)^{12}}{2}\right)$

$= 1 + \frac{1}{\sqrt{3}i}\left(\frac{(-2w^2)^{12} - (2w)^{12}}{2}\right) = 1$

so distance of $(12, \sqrt{3})$ from $x - \sqrt{3}y + 1 = 0$ is

$\frac{12 - 3 + 1}{2} = 5$

23. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{d} = \vec{a} \times \vec{b}$. If $\vec{c}$ is a vector such that $\vec{a}.\vec{c} = |\vec{c}|$, $|\vec{c} - 2\vec{a}|^2 = 8$ and the angle between $\vec{d}$ and $\vec{c}$ is $\frac{\pi}{4}$, then $|10 - 3\vec{b}.\vec{c}| + |\vec{d} \times \vec{c}|^2$ is equal to .....

Ans. (6)

Sol. $\vec{a} = \hat{i} + \hat{j} + \hat{k}$

$\vec{b} = 2\hat{i} + 2\hat{j} + \hat{k}$

$\vec{d} = \vec{a} \times \vec{b}$

$= -\hat{i} + \hat{j}$

$|\vec{c} - 2\vec{a}|^2 = 8$

$|c|^2 + 4|a|^2 - 4(a.c) = 8$

$c^2 + 12 - 4c = 8$

$c^2 - 4c + 4 = 0$

$|c| = 2$

$\vec{d} = \vec{a} \times \vec{b}$

$\vec{d} \times \vec{c} = (\vec{a} \times \vec{b}) \times \vec{c}$

$\left(|a||c|\sin\frac{\pi}{4}\right)^2 = ((a.c).b - (b.c).a)^2$

$4 = 4b^2 + (b.c)2(a^2) - 2(b.c)(a.b)$

$4 = 36 + 3x^2 - 20x$

Let b.c = x

$3x^2 - 20x + 32 = 0$

$3x^2 - 12x - 8x + 32 = 0$

$$x = \frac{8}{3}, 4$$

$$b.c = \frac{8}{3}, 4$$

$$b.c = \frac{8}{3}$$

Now $|10 - 3b.c| + |d \times c|^2$

$|10 - 8| + (2)^2$

$\Rightarrow$ 6 Ans.

24. Let

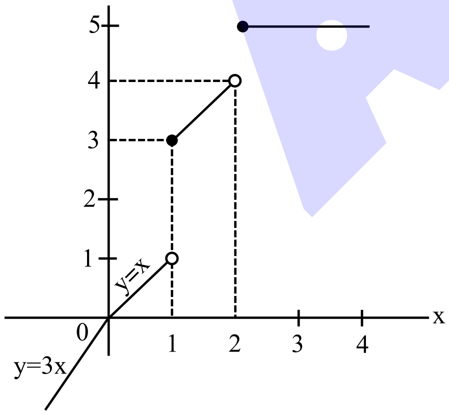
$$f(x) = \begin{cases} 3x, & x < 0 \\ \min\{1 + x + [x], x + 2[x]\}, & 0 \le x \le 2 \\ 5, & x > 2 \end{cases}$$

where [.] denotes greatest integer function. If α and β are the number of points, where f is not continuous and is not differentiable, respectively, then $\alpha + \beta$ equals.......

Ans. **(5)**

Sol.
$$f(x) = \begin{cases} 3x & ; \quad x < 0 \\ \min\{1 + x, x\} & ; \quad 0 \le x < 1 \\ \min\{2 + x, x + 2\} & ; \quad 1 \le x < 2 \\ 5 & ; \quad x > 2 \end{cases}$$

$$f(x) = \begin{cases} 3x & ; \quad x < 0 \\ x & ; \quad 0 \le x < 1 \\ x + 2 & ; \quad 1 \le x < 2 \\ 5 & ; \quad x > 2 \end{cases}$$



Not continuous at $x \in \{1, 2\}$ $\Rightarrow \alpha = 2$

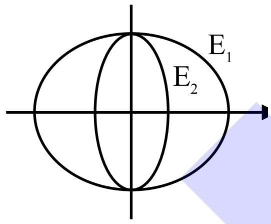
Not diff. at $x \in \{0, 1, 2\}$ $\Rightarrow \beta = 3$

$\alpha + \beta = 5$

25. Let $E_1 : \frac{x^2}{9} + \frac{y^2}{4} = 1$ be an ellipse. Ellipses E_i's are constructed such that their centres and eccentricities are same as that of E_1, and the length of minor axis of E_i is the length of major axis of E_{i+1} $(i \ge 1)$. If A_i is the area of the ellipse E_i, then $\frac{5}{\pi}\left(\sum_{i=1}^{\infty} A_i\right)$, is equal to ......

Ans. **(54)**

Sol.



$$E_1 = \frac{x^2}{9} + \frac{y^2}{4} \Rightarrow e = \sqrt{1 - \frac{4}{9}} = \frac{\sqrt{5}}{3}$$

$$E_2 : \frac{x^2}{a^2} + \frac{y^2}{4} = 1$$

$$e = \frac{\sqrt{5}}{3} = \sqrt{1 - \frac{a^2}{4}} \Rightarrow \frac{5}{9} = 1 - \frac{a^2}{4}$$

$$a^2 = \frac{16}{9}$$

$$E_2 : \frac{x^2}{\frac{16}{9}} + \frac{y^2}{4} = 1$$

$$E_3 : \frac{x^2}{\frac{16}{9}} + \frac{y^2}{b^2} = 1$$

$$e = \frac{\sqrt{5}}{3} = \sqrt{1 - \frac{b^2}{\frac{16}{9}}} \Rightarrow b^2 = \frac{64}{81}$$

$$E_3 = \frac{x^2}{\frac{16}{9}} + \frac{y^2}{\frac{64}{81}} = 1$$

$A_1 = \pi \times 3 \times 2 \Rightarrow 6\pi$

$$A_2 = \pi \times \frac{4}{3} \times 2 = \frac{8\pi}{3}$$

$$A_3 = \pi \times \frac{4}{3} \times \frac{8}{9} = \frac{32\pi}{81}$$

$$\sum_{i=1}^{\infty} A_i = 6\pi + \frac{8\pi}{3} + \frac{32\pi}{81} + ...\infty \Rightarrow \frac{6\pi}{1 - \frac{4}{9}} \Rightarrow \frac{54\pi}{5}$$

$$\therefore \frac{5}{\pi}\sum_{i=1}^{\infty} A_i \Rightarrow \frac{5}{\pi} \times \frac{54\pi}{5} = 54$$

JEE–MAIN EXAMINATION – JANUARY 2025

(HELD ON TUESDAY 28th JANUARY 2025) **TIME : 3:00 PM TO 6:00 PM**

MATHEMATICS — TEST PAPER WITH SOLUTION

SECTION-A

1. Bag B_1 contains 6 white and 4 blue balls, Bag B_2 contains 4 white and 6 blue balls, and Bag B_3 contains 5 white and 5 blue balls. One of the bags is selected at random and a ball is drawn from it. If the ball is white, then the probability, that the ball is drawn from Bag B_2, is :

(1) $\frac{1}{3}$ (2) $\frac{4}{15}$

(3) $\frac{2}{3}$ (4) $\frac{2}{5}$

Ans. (2)

Sol. E_1 : Bag B_1 is selected

B_1	B_2	B_3
6W 4B	4W 6B	5W 5B

E_2 : bag B_2 is selected

E_3 : Bag B_3 is selected

A : Drawn ball is white

We have to find $P\left(\frac{E_2}{A}\right)$

$$P\left(\frac{E_2}{A}\right)=\frac{P(E_2)P\left(\frac{A}{E_2}\right)}{P(E_1)P\left(\frac{A}{E_1}\right)+P(E_2)P\left(\frac{A}{E_2}\right)+P(E_3)P\left(\frac{A}{E_3}\right)}$$

$$=\frac{\frac{1}{3}\times\frac{4}{10}}{\frac{1}{3}\times\frac{6}{10}+\frac{1}{3}\times\frac{4}{10}+\frac{1}{3}\times\frac{5}{10}}$$

$$=\frac{4}{15}$$

2. Let A, B, C be three points in xy-plane, whose position vector are given by $\sqrt{3}\hat{i}+\hat{j}$, $\hat{i}+\sqrt{3}\hat{j}$ and $a\hat{i}+(1-a)\hat{j}$ respectively with respect to the origin O. If the distance of the point C from the line bisecting the angle between the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ is $\frac{9}{\sqrt{2}}$, then the sum of all the possible values of a is :

(1) 1 (2) 9/2

(3) 0 (4) 2

Ans. (1)

Sol. Equation of angle bisector : x – y = 0

$$\left|\frac{a(1-a)}{\sqrt{2}}\right|=\frac{9}{\sqrt{2}}\Rightarrow a=5 \text{ or } -4$$

Sum = 5 + (–4) = 1

3. If the components of $\vec{a}=\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k}$ along and perpendicular to $\vec{b}=3\hat{i}+\hat{j}-\hat{k}$ respectively, are $\frac{16}{11}(3\hat{i}+\hat{j}-\hat{k})$ and $\frac{1}{11}(-4\hat{i}-5\hat{j}-17\hat{k})$, then $\alpha^2+\beta^2+\gamma^2$ is equal to :

(1) 23 (2) 18

(3) 16 (4) 26

Ans. (4)

Sol. let

$\vec{a}_{11}$ = component of $\vec{a}$ along $\vec{b}$

$\vec{a}_1$ = component of $\vec{a}$ perpendicular to $\vec{b}$

$$\vec{a}_{11}=\frac{16}{11}(3\hat{i}+\hat{j}-\hat{k})$$

$$\vec{a}_1=\frac{1}{11}(-4\hat{i}-5\hat{j}-17\hat{k})$$

$\because\ \vec{a}=\vec{a}_{11}+\vec{a}_1$

$$\therefore\ \vec{a}=\frac{16}{11}(3\hat{i}+\hat{j}-\hat{k})+\frac{1}{11}(-4\hat{i}-5\hat{j}-17\hat{k})$$

$$=\frac{44}{11}\hat{i}+\frac{11}{11}\hat{j}-\frac{33}{11}\hat{k}$$

$$\vec{a}=4\hat{i}+\hat{j}-3\hat{k}$$

$\alpha=4$ $\beta=1$ $\gamma=-3$

$\alpha^2+\beta^2+\gamma^2=16+1+9=26$

4. If $\alpha+i\beta$ and $\gamma+i\delta$ are the roots of $x^2-(3-2i)x-(2i-2)=0$, $i=\sqrt{-1}$, then $\alpha\gamma+\beta\delta$ is equal to :

(1) 6 (2) 2

(3) –2 (4) – 6

Ans. (2)

Sol. $x^2 - (3-2i)x - (2i-2) = 0$

$$x = \frac{(3-2i) \pm \sqrt{(3-2i)^2 - 4(1)(-(2i-2))}}{2(1)}$$

$$= = \frac{(3-2i) \pm \sqrt{9-4-12i+8i-8}}{2}$$

$$= = \frac{3-2i \pm \sqrt{-3-4i}}{2}$$

$$= \frac{3-2i \pm \sqrt{(1)^2 + (2i)^2 - 2(1)(2i)}}{2}$$

$$= \frac{3-2i \pm (1-2i)}{2}$$

$$\Rightarrow \frac{3-2i+1-2i}{2} \text{ or } \frac{3-2i-1+2i}{2}$$

$\Rightarrow$ $2-2i$ or $1+0i$

So $\alpha\gamma + \beta\delta = 2(1) + (-2)(0) = 2$

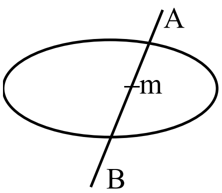
5. If the midpoint of a chord of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ is $(\sqrt{2}, 4/3)$, and the length of the chord is $\frac{2\sqrt{\alpha}}{3}$, then α is :

(1) 18 (2) 22

(3) 26 (4) 20

Ans. **(2)**

Sol.



If $m\left(\sqrt{2}, \frac{4}{3}\right)$ than equation of of AB is

$T = S_1$

$$\frac{x\sqrt{2}}{9} + \frac{y}{4}\left(\frac{4}{3}\right) = \frac{(\sqrt{2})^2}{9} + \frac{\left(\frac{4}{3}\right)^2}{4}$$

$$\frac{\sqrt{2}x}{9} + \frac{y}{3} = \frac{2}{9} + \frac{4}{9}$$

$$\sqrt{2}x + 3y = 6 \Rightarrow y = \frac{6-\sqrt{2}x}{3} \text{ put in ellipse}$$

$$\text{So, } \frac{x^2}{9} + \frac{(6-\sqrt{2}x)^2}{9\times 4} = 1$$

$$4x^2 + 36 + 2x^2 - 12\sqrt{2}x = 36$$

$$6x^2 - 12\sqrt{2}x = 0$$

$$6x\left(x - 2\sqrt{2}\right) = 0$$

$x = 0$ & $x = 2\sqrt{2}$

So $y = 2$ $\quad y = \frac{2}{3}$

$$\text{Length of chord} = \sqrt{\left(2\sqrt{2}-0\right)^2 + \left(\frac{2}{3}-2\right)^2}$$

$$= \sqrt{8 + \frac{16}{9}}$$

$$= \sqrt{\frac{88}{9}} = \frac{2}{3}\sqrt{22} \text{ so } \boxed{\alpha = 22}$$

6. Let S be the set of all the words that can be formed by arranging all the letters of the word GARDEN. From the set S, one word is selected at random. The probability that the selected word will NOT have vowels in alphabetical order is :

(1) $\frac{1}{4}$ (2) $\frac{2}{3}$

(3) $\frac{1}{3}$ (4) $\frac{1}{2}$

Ans. **(4)**

Sol. A, E,G R D N

$$\text{Probabllity (P)} = \frac{\text{favourable case}}{\text{Total case}}$$

(when A & E are in order)

Total case = 6!

Favourable case $= {}^6C_2 \,.\, 4!$

$$P = \frac{(15)4!}{(30)4!}$$

$$\text{Probablity when not in order} = 1 - \frac{1}{2} = \frac{1}{2}$$

7. Let f be a real valued continuous function defined on the positive real axis such that $g(x) = \int_0^x t f(t) dt$.

If $g(x^3) = x^6 + x^7$, then value of $\sum_{r=1}^{15} f(r^3)$ is :

(1) 320 (2) 340

(3) 270 (4) 310

Ans. (4)

Sol. $g(x) = x2 + x^{\frac{7}{3}}$

$g'(x) = 2x + \frac{7}{3}x^{\frac{4}{3}}$

$f(x) = \frac{g'(x)}{x}$

$f(x) = 2 + \frac{7}{3}x^{\frac{1}{3}}$

$f(r^3) = 2 + \frac{7r}{3}$

$\sum_{r=1}^{15}\left(1 + \frac{7}{3}r\right) = 310$

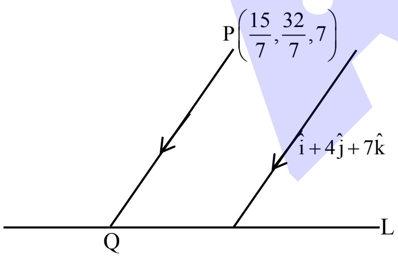
8. The square of the distance of the point $\left(\frac{15}{7}, \frac{32}{7}, 7\right)$ from the line $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ in the direction of the vector $\hat{i} + 4\hat{j} + 7\hat{k}$ is :

(1) 54 (2) 41

(3) 66 (4) 44

Ans. (3)

Sol.



$L = \frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$

$PQ = \frac{x - \frac{15}{7}}{1} = \frac{y - \frac{32}{7}}{4} = \frac{z-7}{7} = \lambda$

$\Rightarrow Q\left(\lambda + \frac{15}{7}, 4\lambda + \frac{32}{7}, 7\lambda + 7\right)$

Since Q lies on line L

So, $\frac{\lambda + \frac{15}{7} + 1}{3} = \frac{7\lambda + 7 + 5}{7}$

$\Rightarrow 7\lambda + 22 = 21\lambda + 36$

$\Rightarrow \lambda = -1$

$\therefore$ Point $Q\left(\frac{8}{7}, \frac{4}{7}, 0\right)$

$PQ = \sqrt{\left(\frac{15}{7} - \frac{8}{7}\right)^2 + \left(\frac{32}{7} - \frac{4}{7}\right)^2 + (7-0)}$

$PQ = \sqrt{66}$

$\Rightarrow (PQ)^2 = 66$

9. The area of the region bounded by the curves $x(1 + y^2) = 1$ and $y^2 = 2x$ is :

(1) $2\left(\frac{\pi}{2} - \frac{1}{3}\right)$ (2) $\frac{\pi}{4} - \frac{1}{3}$

(3) $\frac{\pi}{2} - \frac{1}{3}$ (4) $\frac{1}{2}\left(\frac{\pi}{2} - \frac{1}{3}\right)$

Ans. (3)

Sol. $x(1 + y^2) = 1$ ..... (1)

$y^2 = 2x$ ..... (2)

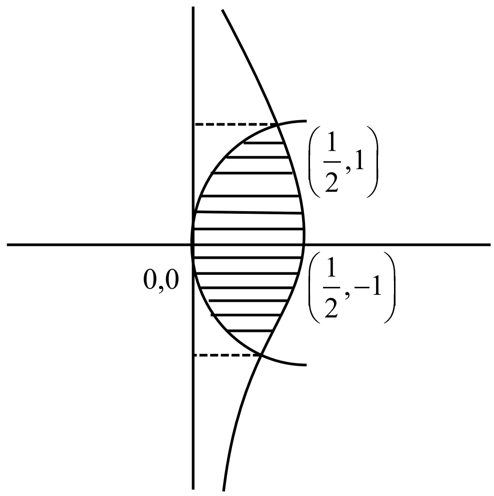
From equation (1) & (2)

$x(1 + 2x) = 1 \Rightarrow 2x^2 + x - 1 = 0$

$\Rightarrow x = \frac{1}{2}$, $x = -1$ (Reject)

$\Rightarrow y^2 = 2\left(\frac{1}{2}\right)$

$\Rightarrow y = \pm 1$



Area bounded $= \int_{-1}^{1}\left(\frac{1}{1+y^2}-\frac{y^2}{2}\right)dy$

$= \left(\tan^{-1}y-\frac{y^3}{6}\right)\Bigg|_{-1}^{1}$

$= \frac{\pi}{2}-\frac{1}{3}$

10. Let $A=\begin{bmatrix}\frac{1}{\sqrt{2}} & -2\\ 0 & 1\end{bmatrix}$ and $P=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$, $\theta>0$.

If $B = PAP^T$, $C = P^TB^{10}P$ and the sum of the diagonal elements of C is $\frac{m}{n}$, where gcd(m, n) = 1, then m + n is :

(1) 65 (2) 127

(3) 258 (4) 2049

Ans. **(1)**

Sol. $P=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$

$\because\ P^TP = I$

$B = PAPT$

Pre multiply by P^T (Given)

$P^TB = P^TP\ AP^T = AP^T$

Now post multiply by P

$P^TBP = AP^TP = A$

So $A^2 = \underbrace{P^TBP\ P^T}_{I}\ BP$

$A^2 = P^TB^2P$

Similarly $A^{10} = P^TB^{10}\ P = C$

$A=\begin{bmatrix}\frac{1}{\sqrt{2}} & -2\\ 0 & 1\end{bmatrix}$ (Given)

$\Rightarrow A^2=\begin{bmatrix}\frac{1}{2} & -\sqrt{2}-2\\ 0 & 1\end{bmatrix}$

Similarly check A^3 and so on since $C = A^{10}$

$\Rightarrow$ Sum of diagonal elements of C is $\left(\frac{1}{\sqrt{2}}\right)^{10}+1$

$= \frac{1}{32}+1=\frac{33}{32}=\frac{m}{n}$

g cd(m,n) = 1 (Given)

$\Rightarrow m + n = 65$

11. If $f(x)=\int\frac{1}{x^{1/4}(1+x^{1/4})}dx$, $f(0) = -6$, then $f(1)$ is equal to :

(1) $\log_e 2+2$ (2) $4(\log_e 2-2)$

(3) $2-\log_e 2$ (4) $4(\log_e 2+2)$

Ans. **(1)**

Sol. let $x = t^4$

$dx = 4t^3\ dt$

then $\int\frac{1}{x^{\frac{1}{4}}\left(1+x^{\frac{1}{4}}\right)}dx \Rightarrow \int\frac{4t^3dt}{t(1+t)}$

$\Rightarrow \int\frac{4t}{1+t}dt \Rightarrow 4\int\frac{(t^2-1)+1}{1+t}dt$

$\Rightarrow 4\int(t-1)+\frac{1}{t+1}dt$

$\Rightarrow 4\left\{\frac{(t-1)^2}{2}+\ell n(t+1)\right\}+c$

hence $f(x) = 2\left(x^{\frac{1}{4}}-1\right)^2+4\ell n\left(1+x^{\frac{1}{4}}\right)+c$

$f(0) = -6 \Rightarrow 2 + 4\ell n + 6 = -6 \rightarrow C = -8$

now $f(1) = 4\ell n\ 2 - 8$

$= 4(\ell n2-2)$

12. Let $f : R \to R$ be a twice differentiable function such that $f(2) = 1$. If $F(x) = xf(x)$ for all $x \in R$, $\int_0^2 xF'(x)dx=6$ and $\int_0^2 x^2F''(x)dx=40$, then

$F'(2) + \int_0^2 F(x)dx$ is equal to :

(1) 11 (2) 15

(3) 9 (4) 13

Ans. **(2)**

Sol. $\int_0^2 xF'(x)dx = 6$

$= xF(x)\Big|_0^2 - \int_0^2 f(x)dx = 6$

$= 2F(2) - \int_0^2 xF(x)dx = 6$ [$\therefore$ f(2) = 2F(2) = 2]

$\int_0^2 xF(x)dx = -2$ ... (1)

$\Rightarrow \int_0^2 F(x)dx = -2$ ... (2)

Also

$\int_0^2 x^2F''(x)dx = x^2F'(x)\Big|_0^2 - 2\int_0^2 xF'(x)dx = 40$

$= 4F'(2) - 2 \times 6 = 40$

$F'(2) = 13$

$\therefore F'(2) + \int_0^2 F(x) = 13 - 2 = 11$

13. For positive integers n, if $4a_n = (n^2 + 5n + 6)$ and $S_n = \sum_{k=1}^{n}\left(\frac{1}{a_k}\right)$, then the value of 507 S_{2025} is :

(1) 540 (2) 1350

(3) 675 (4) 135

Ans. (3)

Sol. $a_n = \frac{n^2 + 5n + 6}{4}$

$S_n = S_n = \sum_{k=1}^{n} \frac{1}{a_k} = \sum_{1}^{n} \frac{4}{k^2 + 5k + 6}$

$= 4\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)}$

$= 4\sum_{k=1}^{n} \frac{1}{k+2} - \frac{1}{k+3}$

$= 4\left(\frac{1}{3} - \frac{1}{4}\right) + 4\left(\frac{1}{4} - \frac{1}{5}\right) + \ldots\ldots$

$4\left(\frac{1}{n+2} - \frac{1}{n+3}\right)$

$= 4\left(\frac{1}{3} - \frac{1}{n+3}\right)$

$= \frac{4n}{3(n+3)}$

$^{507}S_{2025} = \frac{(507)(4)(2025)}{3(2028)}$

$= 675$

14. Let $f : [0, 3] \rightarrow A$ be defined by $f(x) = 2x^3 - 15x^2 + 36x + 7$ and $g : [0, \infty) \rightarrow B$ be defined by $g(x) = \frac{x^{2025}}{x^{2025} + 1}$. If both the functions are onto and $S = \{x \in \mathbf{Z} : x \in A \text{ or } x \in B\}$, then n (S) is equal to :

(1) 30 (2) 36

(3) 29 (4) 31

Ans. (1)

Sol. as f(x) is onto hence A is range of f(x)

now $f'(x) = 6x^2 - 30x + 36$

$= 6(x-2)(x-3)$

$f(2) = 16 - 60 + 72 + 7 = 35$

$f(3) = 54 - 135 + 108 + 7 = 34$

$f(0) = 7$

hence range $\in [7, 35] = A$

also for range of g (x)

$g(x) = 1 - \frac{1}{x^{2025} + 1} \in [0, 1) = B$

s = {0, 7,8, ...35} hence n(s) = 30

15. Let [x] denote the greatest integer less than or equal to x. Then domain of $f(x) = \sec^{-1}(2[x]+1)$ is :

(1) $(-\infty, -1] \cup [0, \infty)$

(2) $(-\infty, -\infty)$

(3) $(-\infty, -1] \cup [1, \infty)$

(4) $(-\infty, \infty] - \{0\}$

Ans. (2)

Sol. $2[x] + 1 \leq -1$ or $2[x] + 1 \geq 1$

$\Rightarrow [x] \leq -1 \cup [x] \geq 0$

$\Rightarrow x \in (-\infty, 0) \cup x \in [0, \infty)$

$\Rightarrow x \in (-\infty, \infty)$

16. If $\sum_{r=1}^{13}\left\{\frac{1}{\sin\left(\frac{\pi}{4}+(r-1)\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}+\frac{r\pi}{6}\right)}\right\}=a\sqrt{3}+b$, $a, b \in \mathbf{Z}$, then a^2+b^2 is equal to :

(1) 10 (2) 2

(3) 8 (4) 4

Ans. **(3)**

Sol. $$\frac{1}{\sin\frac{\pi}{6}}\sum_{r=1}^{13}\frac{\sin\left[\left(\frac{\pi}{4}+\frac{r\pi}{6}\right)-\left(\frac{\pi}{4}\right)-(r-1)\frac{\pi}{6}\right]}{\sin\left(\frac{\pi}{4}+(r-1)\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}+\frac{r\pi}{6}\right)}$$

$$\frac{1}{\sin\frac{\pi}{6}}\sum_{r=1}^{13}\left(\cot\left(\frac{\pi}{4}+(r-1)\frac{\pi}{6}\right)-\cot\left(\frac{\pi}{4}+\frac{r\pi}{6}\right)\right)$$

$$=2\sqrt{3}-2=\alpha\sqrt{3}+b$$

So $a^2+b^2=8$

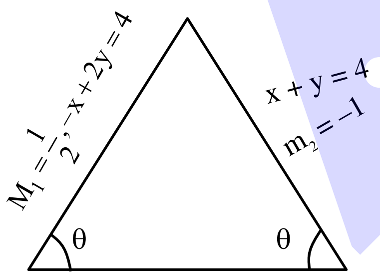
17. Two equal sides of an isosceles triangle are along $-x+2y=4$ and $x+y=4$. If m is the slope of its third side, then the sum, of all possible distinct values of m, is :

(1) –6 (2) 12

(3) 6 (4) $-2\sqrt{10}$

Ans. **(3)**

Sol.



$$\tan\theta=\frac{m-\frac{1}{2}}{1+\frac{1}{2}.m}=\frac{-1-m}{1-m}=\frac{m+1}{m-1}$$

$$\frac{2m-1}{2+m}=\frac{m+1}{m-1}$$

$2m^2-3m+1=m^2+3m+2$

$m^2-6m-1=0$

sum of root = 6

sum is 6

18. Let the coefficients of three consecutive terms T_r, T_{r+1} and T_{r+2} in the binomial expansion of $(a+b)^{12}$ be in a G.P. and let p be the number of all possible values of r. Let q be the sum of all rational terms in the binomial expansion of $\left(\sqrt[4]{3}+\sqrt[3]{4}\right)^{12}$. Then p + q is equal to :

(1) 283 (2) 295

(3) 287 (4) 299

Ans. **(1)**

Sol. $(a+b)^{\frac{1}{2}}$

$T_r, T_{r+1}, T_{r+2} \rightarrow GP$

So, $\frac{T_{r+1}}{T_r}=\frac{T_{r+2}}{T_{r+1}}$

$$\frac{{}^{12}C_r}{{}^{12}C_{r-1}}=\frac{{}^{12}C_{r+1}}{{}^{12}C_r}$$

$$\frac{12-r+1}{r}=\frac{12-(r+1)+1}{r+1}$$

$(13-r)(r+1)=(12-r)(r)$

$-r+12r+13=12r-r^2$

$13=0$

No value of r possible

So P = 0

$$\left(3^{\frac{1}{4}}+4^{\frac{1}{3}}\right)^{12}=\sum {}^{12}C_r\left(3^{\frac{1}{4}}\right)^{12-r}\left(4^{\frac{1}{3}}\right)^r$$

Exponent of $\left(3^{\frac{1}{4}}\right)$	exponent of $\left(4^{\frac{1}{3}}\right)$	term
12	0	27
0	12	256

$q=27+256=283$

$p+q=0+283=283$

19. If A and B are the points of intersection of the circle $x^2 + y^2 - 8x = 0$ and the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ and a point P moves on the line $2x - 3y + 4 = 0$, then the centroid of ΔPAB lies on the line :

(1) $4x - 9y = 12$

(2) $x + 9y = 36$

(3) $9x - 9y = 32$

(4) $6x - 9y = 20$

Ans. **(4)**

Sol. $x^2 + y^2 - 8x = 0, \frac{x^2}{9} - \frac{y^2}{4} = 1$ .... (1)

$4x^2 - 9y^2 = 36$ ... (2)

Solve (1) & (2)

$4x^2 - 9(8x - x^2) = 36$

$13x^2 - 72x - 36 = 0$

$(13x + 6)(x = 6) = 0$

$x = \frac{-6}{13}, x = 6$

$x = \frac{-6}{13}$ (rejected)

$y \rightarrow$ Imaginary

$n = 6, \frac{36}{9} - \frac{y^2}{4} = 1$

$y^2 = 12, y = I\sqrt{12}$

$A(6, \sqrt{12}), B(6, -\sqrt{12})$

$p\left(\alpha, \frac{2\alpha + 4}{3}\right)$ P lies on

centroid (h,k) $2x - 3y + y = 0$

$h = \frac{12 + \alpha}{3}, \alpha = 3h - 12$

$k = \frac{\frac{2\alpha + 4}{3}}{3} \Rightarrow 2\alpha + 4 = 9k$

$\alpha = \frac{9k - 4}{2}$

$6h - 2y = 9k - 4$

$6x - 9y = 20$

20. Let $f : \mathbf{R} - \{0\} \rightarrow (-\infty, 1)$ be a polynomial of degree 2, satisfying $f(x)f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$. If $f(K) = -2K$, then the sum of squares of all possible values of K is :

(1) 1 (2) 6

(3) 7 (4) 9

Ans. **(2)**

Sol. as f(x) is a polynomial of degree two let it be

$f(x) = ax^2 + bx + c \quad (a \neq 0)$

on satisfying given conditions we get

$C = 1$ & $a = \pm 1$

hence $f(x) = 1 \pm x^2$

also range $\in (-\infty, 1]$ hence

$f(x) = 1 - x^2$

now $f(k) = -2k$

$1 - k^2 = -2k \rightarrow k^2 - 2k - 1 = 0$

let roots of this equation be α & β

then $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= 4 - 2(-1) = 6$

SECTION-B

21. The number of natural numbers, between 212 and 999, such that the sum of their digits is 15, is _____.

Ans. **(64)**

Sol.

x	y	z

Let $x = 2 \Rightarrow y + z = 13$

(4,9), (5,8), (6,7), (7,6), (8,5), (9,4), $\rightarrow$ 6

Let $x = 3 \rightarrow y + z = 12$

(3,9), (4,8), ..., (9,3) $\rightarrow$7

Let $x = 4 \rightarrow y + z = 11$

(2,9), (3,8), ..., (9,1) $\rightarrow$9

Let $x = 5 \rightarrow y + z = 10$

(1,9), (2,8), ..., (9,1) $\rightarrow$10

Let $x = 6 \rightarrow y + z = 9$

(0,9), (1,8), ..., (9,0) $\rightarrow$9

Let $x = 7 \rightarrow y + z = 8$

(0,9), (1,7), ..., (8,0) $\rightarrow$9

Let $x = 8 \rightarrow y + z = 7$

(0,7), (1,6), ..., (7,0) $\rightarrow$8

Let $x = 9 \rightarrow y + z = 6$

(0,6), (1,5), ..., (6,0) $\rightarrow$7

Total = 6 = 7 + 8 + 9 + 10 + 9 + 8 + 7 = 64

22. Let $f(x) = \lim_{n\to\infty}\sum_{r=0}^{n}\left(\frac{\tan(x/2^{r+1}) + \tan^3(x/2^{r+1})}{1-\tan^2(x/2^{r+1})}\right)$.

Then $\lim_{x\to 0}\frac{e^x - e^{f(x)}}{(x - f(x))}$ is equal to ____.

Ans. **(1)**

Sol. $f(x) = \lim_{n\to\infty}\sum_{r=0}^{n}\left(\tan\frac{x}{2^r} - \tan\frac{x}{2^{r+1}}\right) = \tan x$

$$\lim_{x\to 0}\left(\frac{e^x - e^{\tan x}}{x - \tan x}\right) = \lim_{x\to 0} e^{\tan x}\frac{\left(e^{x-\tan x} - 1\right)}{\left(x - \tan x\right)}$$

$= 1$

23. The interior angles of a polygon with n sides, are in an A.P. with common difference 6°. If the largest interior angle of the polygon is 219°, then n is equal to _____.

Ans. **(20)**

Sol. $\frac{n}{2}\left(2a + \left(n-1\right)6\right) = \left(n-2\right).180°$

$an + 3n^2 - 3n = (n-2).\ 180°$ ...(1)

Now according to question

$a + (n-1)6° = 219°$

$\Rightarrow a = 225° - 6n°$ ...(2)

Putting value of a from equation (2) in (1)

We get

$(225n - 6n^2) + 3n^2 - 3n = 180n - 360$

$\Rightarrow 2n^2 - 42n - 360 = 0$

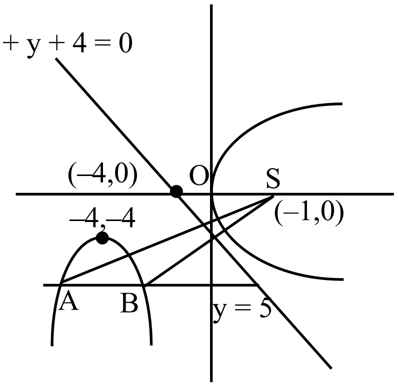
$\Rightarrow n2 - 14n - 120 = 0$

$n = 20, -6$(rejected)

24. Let A and B be the two points of intersection of the line y + 5 = 0 and the mirror image of the parabola $y^2 = 4x$ with respect to the line x + y + 4 = 0. If d denotes the distance between A and B, and a denotes the area of ΔSAB, where S is the focus of the parabola $y^2 = 4x$, then the vlaue of (a + d) is _____.

Ans. **(14)**

Sol.



$\text{Area} = \frac{1}{2}\times 4\times 5 = 10 = a$

$6 = 4$

So $a + d = 14$

25. If y = y(x) is the solution of the differential equation,

$$\sqrt{4-x^2}\frac{dy}{dx} = \left(\left(\sin^{-1}\left(\frac{x}{2}\right)\right)^2 - y\right)\sin^{-1}\left(\frac{x}{2}\right),$$

$-2 \le x \le 2$, $y(2) = \left(\frac{\pi^2 - 8}{4}\right)$, then $y^2(0)$ is equal to ____.

Ans. **(4)**

Sol. $\frac{dy}{dx} + \frac{\left(\sin^{-1}\frac{x}{2}\right)}{\sqrt{4-x^2}}y = \frac{\left(\sin^{-3}\frac{x}{2}\right)^3}{\sqrt{4-x^2}}$

$$y\,e^{\frac{\left(\sin^{-1}\frac{x}{2}\right)^2}{2}} = \int\frac{\left(\sin^{-3}\frac{x}{2}\right)^3}{4-x^2}e^{\frac{\left(\sin^{-1}\frac{x}{2}\right)^2}{2}}dx$$

$$y = \left(\sin^{-1}\frac{x}{2}\right)^2 - 2 + c.e^{-\frac{\left(\sin^{-1}\frac{x}{2}\right)^2}{2}}$$

$y\left(2\right) = \frac{\pi^2}{4} - 2 \Rightarrow c = 0$

$y\ (0) = -2$

JEE–MAIN EXAMINATION – JANUARY 2025

(HELD ON WEDNESDAY 29TH JANUARY 2025) **TIME : 9:00 AM TO 12:00 NOON**

MATHEMATICS — TEST PAPER WITH SOLUTION

SECTION-A

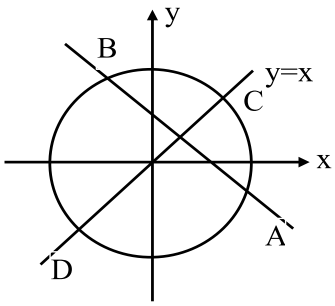
1. Let the line $x + y = 1$ meet the circle $x^2 + y^2 = 4$ at the points A and B. If the line perpendicular to AB and passing through the mid point of the chord AB intersects is the circle at C and D, then the area of the quadrilateral ADBC is equal to

(1) $3\sqrt{7}$ (2) $2\sqrt{14}$

(3) $5\sqrt{7}$ (4) $\sqrt{14}$

Ans. (2)

Sol.



By solving $x = y$ with circle

We get

$C(\sqrt{2},\sqrt{2})$

$D(-\sqrt{2},-\sqrt{2})$

By solving $x + y = 1$ with

circle $x^2 + y^2 = 4$

we set

$A\left(\frac{1+\sqrt{7}}{2},\frac{1-\sqrt{7}}{2}\right)$

$\& \ B\left(\frac{1-\sqrt{7}}{2},\frac{1+\sqrt{7}}{2}\right)$

$\therefore$ Area of Quadrilateral ACBD

$= 2 \times$ Area of ΔBCD

$$= 2\times\frac{1}{2}\begin{vmatrix} \sqrt{2} & \sqrt{2} & 1 \\ \frac{1-\sqrt{7}}{2} & \frac{1+\sqrt{7}}{2} & 1 \\ -\sqrt{2} & -\sqrt{2} & 1 \end{vmatrix}$$

$= 2\sqrt{14}$

2. Let M and m respectively be the maximum and the minimum values of

$$f(x)=\begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 4x \\ \sin^2 x & 1+\cos^2 x & 4\sin 4x \\ \sin^2 x & \cos^2 x & 1+4\sin 4x \end{vmatrix}, x\in R$$

Then $M^4 - m^4$ is equal to :

(1) 1280 (2) 1295

(3) 1040 (4) 1215

Ans. (1)

Sol. $$\begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 4x \\ \sin^2 x & 1+\cos^2 x & 4\sin 4x \\ \sin^2 x & \cos^2 x & 1+4\sin 4x \end{vmatrix}, x\in R$$

$R_2 \to R_2 - R_1 \ \& \ R_3 \to R_3 \to R_1$

$$f(x)\begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 4x \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

Expand about R_1 , use get

$f(x) = 2 + 4\sin 4x$

$\therefore$ M = max value of $f(x) = 6$

M = min value of $f(x) = -2$

$\therefore M^4 - M^4 = 1280$

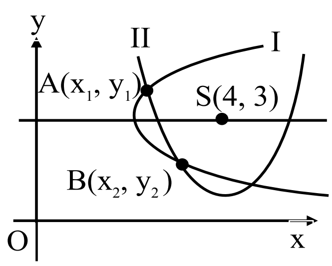
3. Two parabolas have the same focus (4,3) and their directrices are the x-axis and the y-axis, respectively. If these parabolas intersects at the points A and B, then $(AB)^2$ is equal to

(1) 192 (2) 384

(3) 96 (4) 392

Ans. (1)

Sol.



Let intersection points of these two parabolas are $A(x_1, y_1)$ & $B(x_2,y_2)$

$\because$ equation of parabola I and II are given below

$\therefore (x-4)^2 + (y-3)^2 = x^2$ .....(1)

& $(x-4)^2 + (y-3)^2 = y^2$ .....(2)

Here $A(x_1, y_1)$ & $B(x_2,y_2)$ will satisfy with equation

Also from equations (1) & (2), we get = x = y ..(3)

Put x = y in equation (1)

We get $x^2 - 14x + 25 = 0$

$x_1 + x_2 = 14$

$x_1x_2 = 25$

$\therefore AB^2 = (x_1 - x_2)^2 + (y_1-y_2)^2$

$= 2(x_1 - x_2)^2$

$= 2[(x_1+x_2)^2 - 4x_1x_2]$

$= 192$

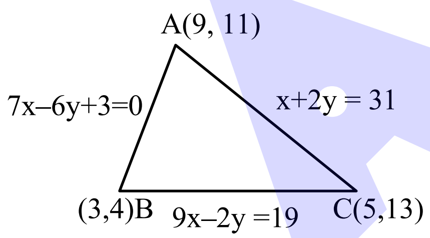
4. Let ABC be a triangle formed by the lines $7x - 6y + 3 = 0$, $x + 2y - 31 = 0$ and $9x - 2y - 19 = 0$, Let the point (h,k) be the image of the centroid of ΔABC in the line $3x + 6y - 53 = 0$. Then $h^2 + k^2 + hk$ is equal to

(1) 37 (2) 47

(3) 40 (4) 36

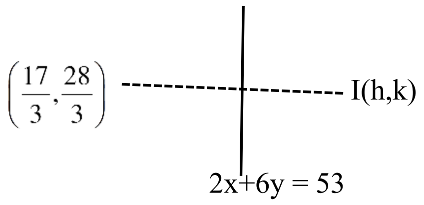
Ans. (1)

Sol.



$\therefore$ centroid of $\Delta ABC = \left(\frac{9+3+5}{3}, \frac{11+4+13}{3}\right)$

$= \left(\frac{17}{3}, \frac{28}{3}\right)$



Let image of centroid with respect to line mirror is (h,k)

$$\therefore \left(\frac{k-\frac{28}{3}}{h-\frac{17}{3}}\right)\left(-\frac{1}{2}\right) = -1$$

$$\& \ 3\left(\frac{h+\frac{17}{3}}{2}\right) + 6.\left(\frac{k+\frac{28}{3}}{2}\right) = 53$$

Solving (1) & (2) we get h = 3, k = 4

$\therefore h^2 + k^2 + hk = 37$

5. Let $\vec{a} = 2\hat{i} - \hat{j} + 3\hat{k}$, $\vec{b} = 3\hat{i} - 5\hat{j} + \hat{k}$ and $\vec{c}$ be a vector such that $\vec{a} \times \vec{c} = \vec{c} \times \vec{b}$ and $(\vec{a}+\vec{c}).(\vec{b}+\vec{c}) = 168$. Then the maximum value of $|\vec{c}|^2$ is :

(1) 77 (2) 462

(3) 308 (4) 154

Ans. (3)

Sol. $\vec{a} = 2\hat{i} - \hat{j} + 3\hat{k}$

$\vec{b} = 3\hat{i} - 5\hat{j} + 3\hat{k}$

$\vec{a} \times \vec{c} = \vec{c} \times \vec{b}$

$\vec{a} \times \vec{c} + \vec{b} \times \vec{c} = 0$

$(\vec{a} + \vec{b}) \times \vec{c} = 0$

$\Rightarrow \vec{c} = \lambda(\vec{a} + \vec{b})$

$\vec{c} = \lambda(5\hat{i} - 6\hat{j} + 4\hat{k})$ .....(1)

$|\vec{c}|^2 = \lambda^2(25 + 36 + 16)$

$|\vec{c}|^2 = 77\lambda^2$

$(\vec{a}+\vec{c}).(\vec{b}+\vec{c}) = 168$

$\vec{a}.\vec{b} + \vec{a}.\vec{c} + \vec{c}.\vec{b} + |\vec{c}|^2 = 168$

$14+\vec{c}.(\vec{a}+\vec{b})+77\lambda^2=168$

using equation (1)

$\lambda\left|5\hat{i}-6\hat{j}+4\hat{k}\right|^2+77\lambda^2=154$

$77\lambda+77\lambda^2-154=0$

$\lambda^2+\lambda-2=0$

$\lambda=-2,\ 1$

$\therefore$ Maximum value of $|\vec{c}|^2$ occurs when $\lambda=-2$

$|\vec{c}|^2=77\lambda^2$

$=77\times4$

$=308$

6. Let P be the set of seven digit numbers with sum of their digits equal to 11. If the numbers in P are formed by using the digits 1,2 and 3 only, then the number of elements in the set P is :

(1) 158 (2) 173

(3) 164 (4) 161

Ans. (4)

Sol. (i) number of numbers created using

$1111133=\frac{7!}{5!2!}\Rightarrow 21$

(ii) number of numbers created using

$1111223=\frac{7!}{4!2!}\Rightarrow 105$

(iii) number of numbers created using

$1112222=\frac{7!}{4!3!}\Rightarrow 35$

Total = 161

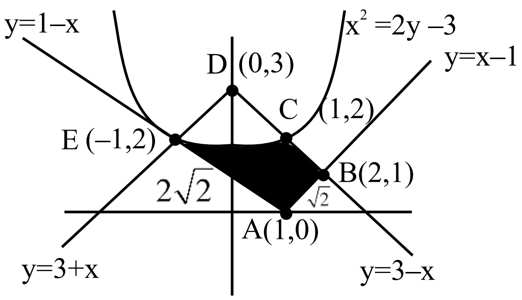
7. Let the area of the region $\{(x,y):2y\le x^2+3,\ y+|x|\le3,\ y\ge|x-1|\}$ be A. Then 6A is equal to:

(1) 16 (2) 12

(3) 18 (4) 14

Ans. (4)

Sol.



A $\Rightarrow$ Rectangle ABDE – Area of region EDC

$A\Rightarrow 4-2\int_0^1(3-x)-\left(\frac{x^2+3}{2}\right)dx$

$A\Rightarrow 4-2\left\{3x-\frac{x^2}{2}-\frac{x^3}{6}-\frac{3}{2}x\right\}_0^1$

$A\Rightarrow 4-2\left\{3-\frac{1}{2}-\frac{1}{6}-\frac{3}{2}\right\}=\frac{7}{3}$

So 6A = 14

8. The least value of n for which the number of integral terms in the Binomial expansion of $\left(\sqrt[3]{7}+\sqrt[12]{11}\right)^n$ is 183, is :

(1) 2184 (2) 2148

(3) 2172 (4) 2196

Ans. (1)

Sol. General term = ${}^nC_r\{7^{1/3}\}^{n-r}\left(11^{1/12}\right)^r$

$={}^nC_r\{7\}^{\frac{n-r}{3}}(11)^{r/12}$

For integral terms, r must be multiple of 12

$\therefore$ r = 12k, k $\in$ W

Total values of r = 183

Hence max r = 12(182)

= 2184

Min value of n = 2184

9. The number of solutions of the equation $\left(\frac{9}{x}-\frac{9}{\sqrt{x}}+2\right)\left(\frac{2}{x}-\frac{7}{\sqrt{x}}+3\right)=0$ is :

(1) 2 (2) 4

(3) 1 (4) 3

Ans. (2)

Sol. Consider $\frac{1}{\sqrt{x}} = \alpha$ $\boxed{x>0}$

$\{9\alpha^2-9\alpha+2\}\{2\alpha^2-7\alpha+3\} = 0$

$(3\alpha-2)(3\alpha-1)(\alpha-3)(2\alpha-1) = 0$

$\alpha = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 3$

$x = 9, 4, \frac{9}{4}, \frac{1}{9}$

So, no. of solutions = 4

10. Let y = y(x) be the solution of the differential equation

$\cos x(\log_e(\cos x))^2 dy + (\sin x - 3y\sin x \log_e(\cos x))dx = 0$,

$x \in \left(0, \frac{\pi}{2}\right)$. If $y\left(\frac{\pi}{4}\right) = \frac{-1}{\log_e 2}$, then $y\left(\frac{\pi}{6}\right)$ is :

(1) $\frac{2}{\log_e(3)-\log_e(4)}$ (2) $\frac{1}{\log_e(4)-\log_e(3)}$

(3) $-\frac{1}{\log_e(4)}$ (4) $\frac{1}{\log_e(3)-\log_e(4)}$

Ans. **(4)**

Sol.

$$\cos x(\ln(\cos x))^2 dy + (\sin x - 3y(\sin x)\ln(\cos x))dx = 0$$

$$\cos x(\ln(\cos x))^2 \frac{dy}{dx} - 3\sin x.\ln(\cos x)y = -\sin x$$

$$\frac{dy}{dx} - \frac{3\tan x}{\ln(\cos x)}y = \frac{-\tan x}{(\ln(\cos x))^2}$$

$$\frac{dy}{dx} + \frac{3\tan x}{\ln(\sec x)}y = \frac{-\tan x}{(\ln(\sec x))^2}$$

$$\text{I.F.} = e^{\int \frac{3\tan x}{\ln(\sec x)}dx} = (\ln(\sec x))^3$$

$$y \times (\ln(\sec x))^3 = -\int \frac{\tan x}{(\ln(\sec x))^2}(\ln(\sec x))^3 dx + C$$

$$y \times (\ln(\sec x))^3 = -\frac{1}{2}(\ln(\sec x))^2 + C$$

Given : $x = \frac{\pi}{4}$, $y = -\frac{1}{\ln 2}$

$$\frac{-1}{\ln 2} \times (\ln\sqrt{2})^3 = -\frac{1}{2} \times (\ln\sqrt{2})^2 + C$$

$$\Rightarrow \frac{-1}{8\ln 2} \times (\ln 2)^3 = \frac{-1}{2} \times \frac{1}{4}(\ln 2)^2 + C$$

$$-\frac{1}{8}(\ln 2)^2 = \frac{-1}{8}(\ln 2)^2 + C$$

$\Rightarrow C = 0$

$$\therefore y(\ln(\sec x))^3 = \frac{-1}{2}(\ln(\sec x))^2 + 0$$

$$y = \frac{-1}{2\ln(\sec x)}$$

$$y = \frac{1}{2\ln(\cos x)}$$

$$\therefore y\left(\frac{\pi}{6}\right) = \frac{1}{2\ln\left(\cos\frac{\pi}{6}\right)}$$

$$= \frac{1}{2\ln\left(\frac{\sqrt{3}}{2}\right)}$$

$$= \frac{1}{2\left(\frac{1}{2}\ln 3 - \ln 2\right)}$$

$$= \frac{1}{\ln 3 - \ln 4}$$

Option (4)

11. Define a relation R on the interval $\left[0, \frac{\pi}{2}\right)$ by x R y if and only if $\sec^2 x - \tan^2 y = 1$. Then R is :

(1) an equivalence relation

(2) both reflexive and transitive but not symmetric

(3) both reflexive and symmetric but not transitive

(4) reflexive but neither symmetric not transitive

Ans. **(1)**

Sol. $\sec^2 x - \tan^2 x = 1$ (on replacing y with x)

$\Rightarrow$ Reflexive

$\sec^2 x - \tan^2 y = 1$

$\Rightarrow 1 + \tan^2 x + 1 - \sec^2 y = 1$

$\Rightarrow \sec^2 y - \tan^2 x = 1$

$\Rightarrow$ symmetric

$\sec^2 x - \tan^2 y = 1$,

$\sec^2 y - \tan^2 z = 1$

Adding both

$\Rightarrow \sec^2x - \tan^2y + \sec^2y - \tan^2 z = 1 + 1$

$\sec^2x + 1 - \tan^2z = 2$

$\sec^2x - \tan^2z = 1$

$\Rightarrow$ Transitive

hence equivalence releation

Option (1)

12. Let the ellipse, $E_1 : \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, a > b and $E_2 : \frac{x^2}{A^2}+\frac{y^2}{B^2}=1$, A < B have same eccentricity $\frac{1}{\sqrt{3}}$. Let the product of their lengths of latus rectums be $\frac{32}{\sqrt{3}}$, and the distance between the foci of E_1 be 4. If E_1 and E_2 meet at A,B,C and D, then the area of the quadrilateral ABCD equals:

(1) $6\sqrt{6}$ (2) $\frac{18\sqrt{6}}{5}$

(3) $\frac{12\sqrt{6}}{5}$ (4) $\frac{24\sqrt{6}}{5}$

Ans. **(4)**

Sol. $2ae = 4$

$2a\left(\frac{1}{\sqrt{3}}\right)=4$

$\Rightarrow a = 2\sqrt{3}$

$\Rightarrow 1-\frac{b^2}{12}=\frac{1}{3}\Rightarrow b^2=8$

Now $\frac{2b^2}{a}\cdot\frac{2A^2}{B}=\frac{32}{\sqrt{3}} \Rightarrow 2\left(\frac{8}{2\sqrt{3}}\right)\frac{2A^2}{B}=\frac{32}{\sqrt{3}}$

$\Rightarrow A^2 = 2B$

$1-\frac{A^2}{B^2}=\frac{1}{3} \Rightarrow 1-\frac{2B}{B^2}=\frac{1}{3} \Rightarrow B = 3$

$\Rightarrow A^2 = 6$

$\frac{x^2}{12}+\frac{y^2}{8}=1$ .....(1)

$\frac{x^2}{6}+\frac{y^2}{9}=1$ .....(2)

On solving (1) & (2) we get

$(x, y) \equiv \left(\frac{\sqrt{6}}{\sqrt{5}},\frac{6}{\sqrt{5}}\right),\left(\frac{-\sqrt{6}}{\sqrt{5}},\frac{6}{\sqrt{5}}\right),\left(\frac{\sqrt{6}}{\sqrt{5}},\frac{-6}{\sqrt{5}}\right),\left(\frac{-\sqrt{6}}{\sqrt{5}},\frac{-6}{\sqrt{5}}\right)$

The four points are vertices of rectangle and its area = $\frac{24\sqrt{6}}{5}$

13. Consider an A.P. of positive integers, whose sum of the first three terms is 54 and the sum of the first twenty terms lies between 1600 and 1800. Then its 11^{th} term is :

(1) 84 (2) 122

(3) 90 (4) 108

Ans. **(3)**

Sol. $S_3 = 3a + 3d = 54$

$\Rightarrow a + d = 18$

$S_{20} = 10(2a + 19d)$

$\Rightarrow 10(36 + 17d)$

$\Rightarrow 1600 < 10(36 + 17d) < 1800$

$\Rightarrow 160 < 36 + 17d < 180$

$\Rightarrow 124 < 17d < 144$

$\Rightarrow 7\frac{5}{17} < d < 8\frac{8}{17}$

Common difference will be natural number

$\Rightarrow d = 8 \Rightarrow a = 10$

$\Rightarrow a_{11} = 10 + 10 \times 8 = 90$

14. Let $\vec{a}=\hat{i}+2\hat{j}+\hat{k}$ and $\vec{b}=2\hat{i}+7\hat{j}+3\hat{k}$. Let $L_1 : \vec{r}=\left(-\hat{i}+2\hat{j}+\hat{k}\right)+\lambda\vec{a}$, $\lambda \in R$ and $L_2 : \vec{r}=\left(\hat{j}+\hat{k}\right)+\mu\vec{b}$, $\mu \in R$ be two lines. If the line L_3 passes through the point of intersection of L_1 and L_2, and is parallel to $\vec{a}+\vec{b}$, then L_3 passes through the point:

(1) (8, 26, 12) (2) (2,8, 5)

(3) (–1, –1, 1) (4) (5, 17, 4)

Ans. **(1)**

Sol. $L_1 : \vec{r} = (-\hat{i}+2\hat{j}+\hat{k})+\lambda(\hat{i}+2\hat{j}+\hat{k})$

$\Rightarrow \vec{r} = (\lambda-1)\hat{i}+2(\lambda+1)\hat{j}+(\lambda+1)\hat{k}$

$L_2 : \vec{r} = (\hat{j}+\hat{k})+\mu(2\hat{i}+7\hat{j}+3\hat{k})$

$\Rightarrow \vec{r} = 2\mu\hat{i}+(1+7\mu)\hat{j}+(1+3\mu)\hat{k}$

For point of intersection equating respective components

$\Rightarrow \lambda - 1 = 2\mu$ ...(1)

$2(\lambda + 1) = 1 + 7\mu$ ....(2)

$\lambda + 1 = 1 + 3\mu$ .....(3)

We get

$\Rightarrow \lambda = 3$ and $\mu = 1$

$\Rightarrow \vec{a}+\vec{b} = 3\hat{i}+9\hat{j}+4\hat{k}$

$L_3 : \vec{r} = 2\hat{i}+8\hat{j}+4\hat{k}+\alpha(3\hat{i}+9\hat{j}+4\hat{k})$

For $\alpha = 2$, $\vec{r} = 8\hat{i}+26\hat{j}+12\hat{k}$

15. The value of $\lim\limits_{n\to\infty}\left(\sum\limits_{K=1}^{n}\frac{k^3+6k^2+11k+5}{(k+3)!}\right)$ is :

(1) $\frac{4}{3}$ (2) 2

(3) $\frac{7}{3}$ (4) $\frac{5}{3}$

Ans. (4)

Sol. $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{k^3+6k^2+11k+5}{(k+3)!}$

$=\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{k^3+6k^2+11k+6-1}{(k+3)!}$

$=\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{(k+1)(k+2)(k+3)-1}{(k+3)!}$

$=\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{(k+1)(k+2)(k+3)}{(k+3)!}-\frac{1}{(k+3)!}$

$=\lim\limits_{k=1}\sum\limits_{k=1}^{n}\left(\frac{1}{k!}-\frac{1}{(k+3)!}\right)$

$=\lim\limits_{k=1}\left(\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}....+\frac{1}{n!}-\frac{1}{4!}-\frac{1}{5!}-\frac{1}{6!}....-\frac{1}{(n+3)!}\right)$

$=\frac{1}{1}+\frac{1}{2}+\frac{1}{6}=\frac{10}{6}=\frac{5}{3}$

16. The integral $80\int_0^{\frac{\pi}{4}}\left(\frac{\sin\theta+\cos\theta}{9+16\sin 2\theta}\right)d\theta$ is equal to :

(1) $3\log_e 4$ (2) $6\log_e 4$

(3) $4\log_e 3$ (4) $2\log_e 3$

Ans. (3)

Sol. $I = 80\int_0^{\frac{\pi}{4}}\left(\frac{\sin\theta+\cos\theta}{9+16(2\sin\theta.\cos\theta)}\right)d\theta$

$=80\int_0^{\frac{\pi}{4}}\frac{\sin\theta+\cos\theta}{9-16(1-2\sin\theta.\cos\theta-1)}d\theta$

$= =80\int_0^{\frac{\pi}{4}}\frac{\sin\theta+\cos\theta}{9+16-16(\sin\theta-\cos\theta)^2}d\theta$

Let $\sin\theta - \cos\theta = t$

$(\cos\theta + \sin\theta)d\theta = dt$

$=80\int_{-1}^{0}\frac{dt}{25-16t^2}$

$=\frac{80}{16}\int_{-1}^{0}\frac{dt}{\left(\frac{5}{4}\right)^2-t^2}$

$=\frac{5}{2\left(\frac{5}{4}\right)}\ln\left[\left(\frac{\frac{5}{4}+t}{\frac{5}{4}-t}\right)\right]_{-1}^{0}$

$= 2\ln(1) + 4\ln 3$

$= 4\ln 3$

17. Let $L_1 : \frac{x-1}{1}=\frac{y-2}{-1}=\frac{z-1}{2}$ and

$L_2 : \frac{x+1}{-1}=\frac{y-2}{2}=\frac{z}{1}$ be two lines.

Let L_3 be a line passing through the point (α,β,γ) and be perpendicular to both L_1 and L_2. If L_3 intersects L_1, then $|5\alpha-11\beta-8\gamma|$ equals :

(1) 18 (2) 16

(3) 25 (4) 20

Ans. (3)

Sol. DR's of $L_3 = \vec{m}\times\vec{n} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 2\\ -1 & 2 & 1\end{vmatrix}$

$=-5\hat{i}-3\hat{j}+\hat{k}$

$L_3 : \frac{x-\alpha}{-5}=\frac{y-\beta}{-3}=\frac{z-\gamma}{1}=\lambda$

$A(\alpha-5\lambda, \beta-3\lambda, \gamma+\lambda)$

$L_1 : \frac{x-1}{1}=\frac{y-2}{-1}=\frac{z-1}{2}=k$

B(k + 1, –k + 2, 2k + 1)

Now

$\alpha - 5\lambda = k+1 \Rightarrow \alpha = 5\lambda + k + 1$

$\beta - 3\lambda = -k+2 \Rightarrow \beta = 3\lambda - k + 2$

$\gamma + \lambda = 2k-1 \Rightarrow \gamma = -\lambda + 2k + 1$

$|5\alpha - 11\beta - 8\gamma| = |-25|$

$= 25$

18. Let $x_1, x_2, ...x_{10}$ be ten observations such that $\sum_{i=1}^{10}(x_i - 2) = 30$, $\sum_{i=1}^{10}(x_i - \beta)^2 = 98$, $\beta > 2$ and their variance is $\frac{4}{5}$. If μ and σ^2 are respectively the mean and the variance of $2(x_1-1) + 4\beta$, $2(x_2-1) + 4\beta$, ..., $2(x_{10}-1) + 4\beta$, then $\frac{\beta\mu}{\sigma^2}$ is equal to :

(1) 100 (2) 110

(3) 120 (4) 90

Ans. (1)

Sol. $\frac{4}{5} = \frac{\Sigma x_i^2}{10} - \left(\frac{\Sigma x_i}{10}\right)^2$

$\frac{4}{5} = \frac{\Sigma x_i^2}{10} - 25$

$\Rightarrow \Sigma x_i^2 = 258$

Now $\sum_{i=1}^{10}(x_i - \beta)^2 = 98$

$\sum_{i=1}^{10}\left(x_i^2 - 2\beta . x_i + \beta^2\right) = 98$

$258 - 2\beta(50) + 10\beta^2 = 98$

$(\beta - 8)(\beta - 2) = 0$

$\beta =$ or $\beta = 2$ (as $\beta > 2$)

$\therefore \beta = 8$

Now,

$= 2(x_1-1) + 4\beta, 2(x_2-1) + 4\beta, ...2(x_{10}-1) + 4\beta$

$= 2x_1 + 30, 2x_2 + 30, ...2x_{10} + 30$

$\mu = 2(5) + 30 = 40$

$\sigma^2 = 2^2\left(\frac{4}{5}\right) = \frac{16}{5}$

$\because \frac{B\mu}{\sigma^2} = \frac{8 \times 40}{16/5} = 100$

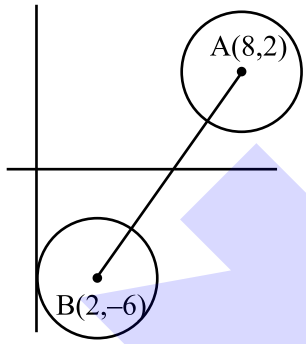
19. Let $|z_1 - 8 - 2i| \le 1$ and $|z_2 - 2 + 6i| \le 2$, $z_1, z_2 \in C$. Then the minimum value of $|z_1 - z_2|$ is :

(1) 3 (2) 7

(3) 13 (4) 10

Ans. (2)

Sol.



$\because AB = \sqrt{100} = 10$

$\therefore |Z_1 - Z_2|_{min} = 10 - 2 - 1 = 7$

20. Let $A = [a_{ij}] = \begin{bmatrix} \log_5 128 & \log_4 5 \\ \log_5 8 & \log_4 25 \end{bmatrix}$.

If A_{ij} is the cofactor of a_{ij}, $C_{ij} = \sum_{k=1}^{2} a_{ik} A_{jk}$, $1 \le i, j \le 2$, and $C = [C_{ij}]$, then $8|C|$ is equal to :

(1) 262 (2) 288

(3) 242 (4) 222

Ans. (3)

$|A| = \frac{11}{2}$

$C_{11} = \sum_{k=1}^{2} a_{1k}.A_{1k} = a_{11}A_{11} + a_{12}A_{12} = |A| = \frac{11}{2}$

$C_{12} = \sum_{k=1}^{2} a_{1k}.A_{2k} = 0$

$C_{21} = \sum_{k=1}^{2} a_{2k}.A_{1k} = 0$

$C_{22} = \sum_{k=1}^{2} a_{2k}.A_{2k} = |A| = \frac{11}{2}$

$C = \begin{bmatrix} 11/2 & 0 \\ 0 & 11/2 \end{bmatrix}$

$|C| = \frac{121}{4}$

$8|C| = 242$

SECTION-B

21. Let $f : (0,\infty) \rightarrow R$ be a twice differentiable function. If for some $a \neq 0$, $\int_0^1 f(\lambda x)d\lambda = af(x)$, $f(1) = 1$ and $f(16) = \frac{1}{8}$, then $16 - f'\left(\frac{1}{16}\right)$ is equal to ______,

Ans. **(112)**

Sol. $\int_0^1 f(\lambda x)d\lambda = af(x)$

$\lambda x = t$

$d\lambda = \frac{1}{x}dt$

$\frac{1}{x}\int_0^x f(t)dt = af(x)$

$\int_0^x f(t)dt = axf(x)$

$f(x) = a(x f'(x) + f(x))$

$(1 - a)f(x) = a.x f'(x)$

$\frac{f'(x)}{f(x)} = \frac{(1-a)}{a}\frac{1}{x}$

$\ell nf(x) = \frac{1-a}{a}\ \ell nx + c$

$x = 1, f(1) = 1 \Rightarrow c = 0$

$x = 16, f(16) = \frac{1}{8}$

$\frac{1}{8} = (16)^{\frac{1-a}{a}} \Rightarrow -3 = \frac{4-4a}{a} \Rightarrow a = 4$

$f(x) = x^{-\frac{3}{4}}$

$f'(x) = -\frac{3}{4}x^{-\frac{7}{4}}$

$\therefore\ 16 - f'\left(\frac{1}{16}\right)$

$= 16 - \left(-\frac{3}{4}(2^{-4})^{-7/4}\right)$

$= 16 + 96 = 112$

22. Let $S = \left\{m \in Z : A^{m^2} + A^m = 3I - A^{-6}\right\}$, where $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$. Then n(S) is equal to ____.

Ans. **(2)**

Sol. $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$

$A^2 = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}, A^3 = \begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}, A^4 = \begin{bmatrix} 5 & -4 \\ 4 & -3 \end{bmatrix}$

and so on

$A^6 = \begin{bmatrix} 7 & -6 \\ 6 & -5 \end{bmatrix}$

$A^m = \begin{bmatrix} m+1 & -m \\ m & -m-1 \end{bmatrix}$,

$A^{m^2} = \begin{bmatrix} m^2+1 & -m^2 \\ m^2 & -(m^2-1) \end{bmatrix}$

$A^{m^2} + A^m = 3I - A^{-6}$

$\begin{bmatrix} m+1 & -m^2 \\ m^2 & -(m^2-1) \end{bmatrix} + \begin{bmatrix} m+1 & -m \\ m & -(m-1) \end{bmatrix}$

$= 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -5 & 6 \\ -6 & 7 \end{bmatrix}$

$= \begin{bmatrix} 8 & -6 \\ 6 & -4 \end{bmatrix}$

$= m^2 + 1 + m + 1 = 8$

$= m^2 + m - 6 = 0 \Rightarrow m = -3, 2$

$n(s) = 2$

23. Let [t] be the greatest integer less than or equal to t. Then the least value of $p \in N$ for which

$$\lim_{x\to 0^+}\left(x\left(\left[\frac{1}{x}\right]+\left[\frac{2}{x}\right]+.....+\left[\frac{p}{x}\right]\right)-x^2\left(\left[\frac{1}{x^2}\right]+\left[\frac{2^2}{x^2}\right]+....+\left[\frac{9^2}{x^2}\right]\right)\right)\geq 1$$

is equal to ____.

Ans. **(24)**

Sol. $\lim_{x\to 0^+}\left(x\left(\left[\frac{1}{x}\right]+\left[\frac{2}{x}\right]+.....+\left[\frac{p}{x}\right]\right)-x^2\left(\left[\frac{1}{x^2}\right]+\left[\frac{2^2}{x^2}\right]+\left[\frac{9^2}{x^2}\right]\right)\right) \geq 1$

$(1 + 2 + ...+ p) - (1^2 + 2^2 + ...9^2) \geq 1$

$$\frac{p(p+1)}{2}-\frac{9.10.19}{6}\geq 1$$

$p\ (p + 1)\ \geq 572$

Least natural value of p is 24

24. The number of 6-letter words, with or without meaning, that can be formed using the letters of the word MATHS such that any letter that appears in the word must appear at least twice, is 4 ____.

Ans. **(1405)**

Sol. (i) Single letter is used , then no. of words = 5

(ii) Two distinct letters are used, then no. of words

$$^5C_2\times\left(\frac{6!}{2!4!}\times 2+\frac{6!}{3!3!}\right)=10(30+20)=500$$

(iii) Three distinct letters are used, then no. of words

$$^5C_3\times\frac{6!}{2!2!2!}=900$$

Total no. of words = 1405

25. Let $S=\left\{x:\cos^{-1}x=\pi+\sin^{-1}x+\sin^{-1}(2x+1)\right\}$.

Then $\sum\limits_{x\in S}(2x-1)^2$ is equal to ______.

Ans. **(5)**

Sol. $\cos^{-1}x = \pi + \sin^{-1}x + \sin^{-1}(2x + 1)$

$2\cos^{-1}x-\sin^{-1}(2x + 1) = \dfrac{3\pi}{2}$

$2\alpha - \beta = \dfrac{3\pi}{2}$ where $\cos^{-1}x = \alpha$, $\sin^{-1}(2x+1) = \beta$

$2\alpha = \dfrac{3\pi}{2} + \beta$

$\cos 2\alpha = \sin\beta$

$2\cos^2\alpha - 1 = \sin\beta$

$2x^2 - 1 = 2x + 1$

$x^2 - x - 1 = 0$

$$\Rightarrow n=\frac{1\pm\sqrt{5}}{2}=\begin{cases} n=\dfrac{1+\sqrt{5}}{2} & \text{rejedcted} \\ n=\dfrac{1-\sqrt{5}}{2} & \end{cases}$$

$\therefore 4x^2 - 4x = 4$

$(2x-1)^2 = 5$

JEE–MAIN EXAMINATION – JANUARY 2025

(HELD ON WEDNESDAY 29th JANUARY 2025) **TIME : 3:00 PM TO 6:00 PM**

MATHEMATICS — TEST PAPER WITH SOLUTION

SECTION-A

1. If the set of all $a \in \mathbf{R}$, for which the equation $2x^2 + (a-5)x + 15 = 3a$ has no real root, is the interval (α,β), and $X = \{x \in Z : \alpha < x < \beta\}$, then $\sum_{x \in X} x^2$ is equal to

(1) 2109 (2) 2129

(3) 2139 (4) 2119

Ans. (3)

Sol. $(a-5)^2 - 8(15-3a) < 0$

$a^2 + 14a + 25 - 120 < 0$

$a^2 + 14a - 95 < 0$

$(a+19)(a-5) < 0$

$a \in (-19, 5)$

$\therefore -19 < x < 5$

$\therefore \sum_{x \in X} x^2 = (1^2 + 2^2 + \ldots. + 4^2) + (1^2 + 2^2 + \ldots + 18^2)$

$= \dfrac{4\times5\times9}{6} + \dfrac{18\times19\times37}{6}$

$= 30 + 2109$

$= 2139$

2. If $\sin x + \sin^2 x = 1$, $x \in \left(0, \dfrac{\pi}{2}\right)$, then

$(\cos^{12}x + \tan^{12}x) + 3(\cos^{10}x + \tan^{10}x + \cos^8 x + \tan^8 x) + (\cos^6 x + \tan^6 x)$ is equal to

(1) 4 (2) 3

(3) 2 (4) 1

Ans. (3)

Sol. $\sin x + \sin^2 x = 1$

$\Rightarrow \sin x = \cos^2 x \Rightarrow \tan x = \cos x$

$\therefore$ Given expression

$= 2\cos^{12}x + 6[\cos^{10}x + \cos^8 x] + 2\cos^6 x$

$= 2[\sin^6 x + 3\sin^5 x + 3\sin^4 x + \sin^3 x]$

$= 2\sin^3 x[(\sin x + 1)^3]$

$= 2[\sin^2 x + \sin x]^3$

$= 2$

3. Let the area enclosed between the curves $|y| = 1 - x^2$ and $x^2 + y^2 = 1$ be α. If $9\alpha = \beta\pi + \gamma$; β, γ are integers, then the value of $|\beta - \gamma|$ equals

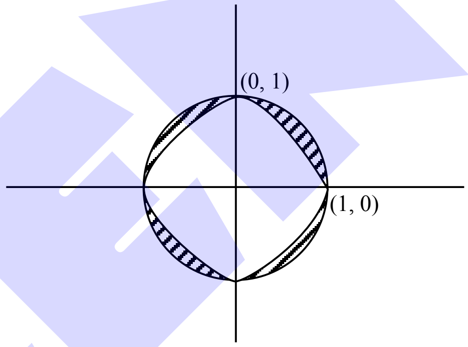
(1) 27 (2) 18

(3) 15 (4) 33

Ans. (4)

Sol. $C_1 : |y| = 1 - x^2$

$C_2 : x^2 + y^2 = 1$



$\therefore$ Required Area

$= \alpha = 4[\text{Area of circle in } 1^{st} \text{ quad.} - \int_0^1 (1-x^2)dx\,]$

$= 4\left[\dfrac{\pi}{4} - \left[x - \dfrac{x^3}{3}\right]_0^1\right]$

$\alpha = \pi - \dfrac{8}{3}$

$\therefore 3\alpha = 3\pi - 8$

$\therefore 9\alpha = 9\pi - 24$

$\therefore \beta = 9, \gamma = -24$

$\therefore |\beta - \gamma| = 33$

4. If the domain of the function $\log_5\left(18x - x^2 - 77\right)$ is (α,β) and the domain of the function $\log_{(x-1)}\left(\dfrac{2x^2+3x-2}{x^2-3x-4}\right)$ is (γ,δ), then $\alpha^2 + \beta^2 + \gamma^2$ is equal to :

(1) 195 (2) 174

(3) 186 (4) 179

Ans. **(3)**

Sol. $f_1(x) = \log_5(18x - x^2 - 77)$

$\therefore\ 18x - x^2 - 77 > 0$

$x^2 - 18x + 77 < 0$

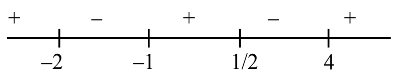
$x \in (7, 11)\quad \alpha = 7, \beta = 11$

$f_2(x) = \log_{(x-1)}\left(\dfrac{2x^2+3x-2}{x^2-3x-4}\right)$

$\therefore\ x - 1 > 0\ ,\ x - 1 \neq 1,\ \dfrac{2x^2+3x-2}{x^2-3x-4} > 0$

$x > 1\ ,\ x \neq 2\ ,\ \dfrac{(2x-1)(x+2)}{(x-4)(x+1)} > 0$

$x > 1\ ,\ x \neq 2\ ,$



$\therefore\ x \in (4, \infty)$

$\therefore\ \gamma = 4$

$\therefore\ \alpha^2 + \beta^2 + \gamma^2 = 49 + 121 + 16$

$= 186$

5. Let the function $f(x) = (x^2 - 1)|x^2 - ax + 2| + \cos|x|$ be not differentiable at the two points $x = \alpha = 2$ and $x = \beta$. Then the distance of the point (α,β) from the line $12x + 5y + 10 = 0$ is equal to :

(1) 3 (2) 4

(3) 2 (4) 5

Ans. **Allen Ans. (BONUS)**

NTA Ans. (1)

Sol. $\cos|x|$ is always differentiable

$\therefore$ we have to check only for $|x^2 - ax + 2|$

$\therefore$ Not differentiable at

$x^2 - ax + 2 = 0$

One root is given, $\alpha = 2$

$\therefore\ 4 - 2a + 2 = 0$

$a = 3$

$\therefore$ other root $\beta = 1$

but for $x = 1$ $f(x)$ is differentiable

(Drop)

6. Let a straight line L pass through the point P(2,–1,3) and be perpendicular to the lines $\dfrac{x-1}{2} = \dfrac{y+1}{1} = \dfrac{z-3}{-2}$ and $\dfrac{x-3}{1} = \dfrac{y-2}{3} = \dfrac{z+2}{4}$.

If the line L intersects the yz-plane at the point Q, then the distance between the points P and Q is :

(1) 2 (2) $\sqrt{10}$

(3) 3 (4) $2\sqrt{3}$

Ans. **(3)**

Sol. Vector parallel to 'L'

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 1 & 3 & 4 \end{vmatrix} = 10\hat{i} - 10\hat{j} + 5\hat{k}$$

$= 5(2\hat{i} - 2\hat{j} + \hat{k})$

Equation of 'L'

$\dfrac{x-2}{2} = \dfrac{y+1}{-2} = \dfrac{z-3}{1} = \lambda$ (say)

Let $Q(2\lambda + 2, -2\lambda - 1, \lambda + 3)$

$\Rightarrow 2\lambda + 2 = 0 \ \Rightarrow \lambda = -1$

$\Rightarrow Q\ (0, 1, 2)$

$d(P, Q) = 3$

7. Let $S = \mathbf{N} \cup \{0\}$. Define a relation **R** from S to **R** by :

$$\mathbf{R} = \left\{(x,y) : \log_e y = x\log_e\left(\frac{2}{5}\right), x \in S, y \in \mathbf{R}\right\}.$$

Then, the sum of all the elements in the range of **R** is equal to

(1) $\dfrac{3}{2}$ (2) $\dfrac{5}{3}$

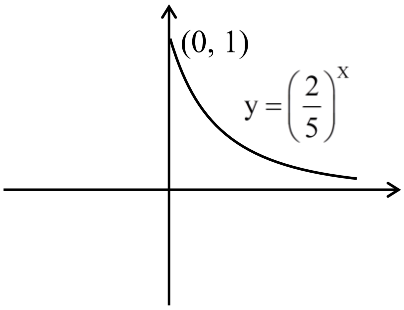
(3) $\dfrac{10}{9}$ (4) $\dfrac{5}{2}$

Ans. **(2)**

Sol. $S = \{0, 1, 2, 3 \ldots..\}$

$\log_e y = x\log_e\left(\dfrac{2}{5}\right)$

$\Rightarrow\ y = \left(\dfrac{2}{5}\right)^x$



Required

$$\text{Sum} = 1+\left(\frac{2}{5}\right)^1+\left(\frac{2}{5}\right)^2+\left(\frac{2}{5}\right)^3+.....-=\frac{1}{1-\frac{2}{5}}=\frac{5}{3}$$

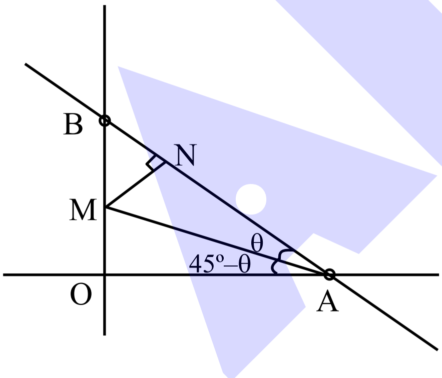
8. Let the line x + y = 1 meet the axes of x and y at A and B, respectively. A right angled triangle AMN is inscribed in the triangle OAB, where O is the origin and the points M and N lie on the lines OB and AB, respectively. If the area of the triangle AMN is $\frac{4}{9}$ of the area of the triangle OAB and AN : NB = λ : 1, then the sum of all possible value(s) of is λ :

(1) $\frac{1}{2}$ (2) $\frac{13}{6}$

(3) $\frac{5}{2}$ (4) 2

Ans. (4)

Sol.



Area of $\Delta AOB = \frac{1}{2}$

Area of $\Delta AMN = \frac{4}{9}\times\frac{1}{2}=\frac{2}{9}$

Equation of AB is x + y = 1

OA = 1, AM = sec(45° – θ)

AN = sec(45° – θ) cosθ

MN = sec(45° – θ) sinθ

$$\text{Ar}(\Delta AMN)=\frac{1}{2}\times \sec^2(45^\circ-\theta)\sin\theta.\cos\theta=\frac{2}{9}$$

$$\Rightarrow \tan\theta = 2, \frac{1}{2}$$

tanθ = 2 is rejected

$$\frac{AN}{NB}=\frac{\lambda}{1}=\cot\theta=2$$

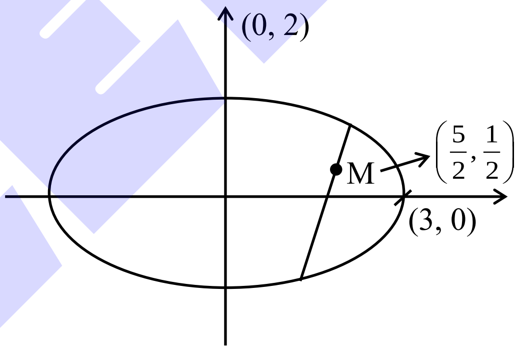
9. If αx + βy = 109 is the equation of the chord of the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$, whose mid point is $\left(\frac{5}{2},\frac{1}{2}\right)$, then α + β is equal to

(1) 37 (2) 46

(3) 58 (4) 72

Ans. (3)

Sol.



Equation of chord T = S_1

$$\frac{5}{2}\left(\frac{x}{9}\right)+\frac{1}{2}\left(\frac{y}{4}\right)=\frac{25}{36}+\frac{1}{16}$$

$$\Rightarrow \frac{5x}{18}+\frac{y}{8}=\frac{100+9}{144}=\frac{109}{144}$$

⇒ 40x + 18y = 109

⇒ α = 40, β = 18

⇒ α + β = 58

10. If all the words with or without meaning made using all the letters of the word "KANPUR" are arranged as in a dictionary, then the word at 440^{th} position in this arrangement, is :

(1) PRNAKU (2) PRKANU

(3) PRKAUN (4) PRNAUK

Ans. (3)

Sol. A, K, N, P, R, U

A .... = $\lfloor 5$ = 120

K .... = $\lfloor 5$ = 120

N .... = $\lfloor 5$ = 120

P A .... = $\lfloor 4$ = 24

P K .... = $\lfloor 4$ = 24

P N .... = $\lfloor 4$ = 24

P R A .... = $\lfloor 3$ = 6

P R K A N U = 1

P R K A U N = 1

Total = 440

$\Rightarrow$ 440[th] word

11. Let α,β $(\alpha \neq \beta)$ be the values of m, for which the equations $x + y + z = 1$; $x + 2y + 4z = m$ and $x + 4y + 10z = m^2$ have infinitely many solutions. Then the value of $\sum_{n=1}^{10}\left(n^{\alpha}+n^{\beta}\right)$ is equal to :

(1) 440 (2) 3080

(3) 3410 (4) 560

Ans. (1)

Sol. $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{vmatrix} = 1(20-16)-1(10-4)+1(4-2)$

$= 4-6+2 = 0$

For infinite solutions

$\Delta_x = \Delta_y = \Delta_z = 0$

$m^2 - 3x + 2 = 0$

$m = 1, 2$

$\alpha = 1, \beta = 2$

$\therefore \sum_{n=1}^{10}(n^{\alpha}+n^{\beta}) = \sum_{n=1}^{10} n^1 + \sum_{n=1}^{10} n^2$

$= \frac{10(11)}{2} + \frac{10(11)(21)}{6}$

$= 55 + 385$

$= 440$

12. Let $A = [a_{ij}]$ be a matrix of order 3 × 3, with $a_{ij} = \left(\sqrt{2}\right)^{i+j}$. If the sum of all the elements in the third row of A^2 is $\alpha + \beta\sqrt{2}$, $\alpha,\beta \in \mathbf{Z}$, then $\alpha + \beta$ is equal to

(1) 280 (2) 168

(3) 210 (4) 224

Ans. (4)

Sol. $A = \begin{bmatrix} (\sqrt{2})^2 & (\sqrt{2})^3 & (\sqrt{2})^4 \\ (\sqrt{2})^3 & (\sqrt{2})^4 & (\sqrt{2})^5 \\ (\sqrt{2})^4 & (\sqrt{2})^5 & (\sqrt{2})^6 \end{bmatrix}$

$A = \begin{bmatrix} 2 & 2\sqrt{2} & 4 \\ 2\sqrt{2} & 4 & 4\sqrt{2} \\ 4 & 4\sqrt{2} & 8 \end{bmatrix}$

$A^2 = 2^2\begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 2 & 2\sqrt{2} \\ 2 & 2\sqrt{2} & 4 \end{bmatrix}\begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 2 & 2\sqrt{2} \\ 2 & 2\sqrt{2} & 4 \end{bmatrix}$

$= 4\begin{bmatrix} - & - & - \\ - & - & - \\ (2+4+8) & (2\sqrt{2}+4\sqrt{2}+8\sqrt{2}) & (4+8+16) \end{bmatrix}$

Sum of elements of 3[rd] row = $4(14 + 14\sqrt{2} + 28)$

$= 4(42 + 14\sqrt{2})$

$= 168 + 56\sqrt{2}$

$\alpha + \beta\sqrt{2}$

$\therefore$ $\alpha + \beta = 168 + 56 = 224$

13. Let P be the foot of the perpendicular from the point (1, 2, 2) on the line L : $\frac{x-1}{1} = \frac{y+1}{-1} = \frac{z-2}{2}$.

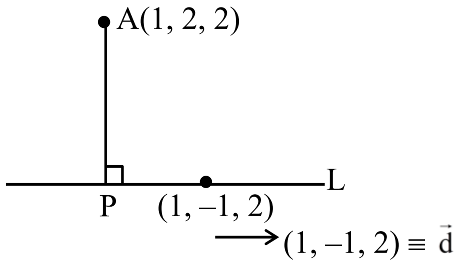
Let the line $\vec{r} = \left(-\hat{i}+\hat{j}-2\hat{k}\right)+\lambda\left(\hat{i}-\hat{j}+\hat{k}\right)$, $\lambda \in \mathbf{R}$, intersect the line L at Q. Then $2(PQ)^2$ is equal to:

(1) 27 (2) 25

(3) 29 (4) 19

Ans. (1)

Sol.



$$L : \frac{x-1}{1} = \frac{y+1}{-1} = \frac{z-2}{2} = \mu$$

$P(\mu + 1, - \mu - 1, 2\mu + 2)$

$\overrightarrow{AP} \cdot \vec{d} = 0 \Rightarrow (\mu, -\mu - 3, 2\mu) \cdot (1, -1, 2) = 0$

$\Rightarrow \mu + \mu + 3 + 4\mu = 0 \Rightarrow \mu = -\frac{1}{2}$

$\therefore P\left(\frac{-1}{2} + 1, +\frac{1}{2} - 1, 2\left(\frac{-1}{2}\right) + 2\right)$

$P\left(\frac{1}{2}, \frac{-1}{2}, 1\right)$

Now general pt. on L_2 is $Q(-1 + \lambda, 1 - \lambda, -2 + \lambda)$

Equate it with general pt of L

$\mu + 1 = -1 + \lambda$	$-\mu - 1 = 1 - \lambda$	$2\mu + 2 = -2 + \lambda$
$\mu = \lambda - 2$	$\mu = \lambda - 2$	↓

$2(\lambda - 2) + 2 = -2 + \lambda$

$2\lambda - 4 + 2 = -2 + \lambda$

$\therefore \mu = -2, \lambda = 0$

$\therefore Q \equiv (-1, 1 -2)$

$P\left(\frac{1}{2}, \frac{-1}{2}, 1\right)$ and $Q(-1, 1, -2)$

$$PQ = \sqrt{\left(\frac{1}{2} + 1\right)^2 + \left(\frac{-1}{2} - 1\right)^2 + (1 + 2)^2}$$

$$= \sqrt{\frac{9}{4} + \frac{9}{4} + 9} = \sqrt{\frac{54}{4}}$$

$$\therefore 2(PQ)^2 = 2\left(\frac{54}{4}\right) = 27$$

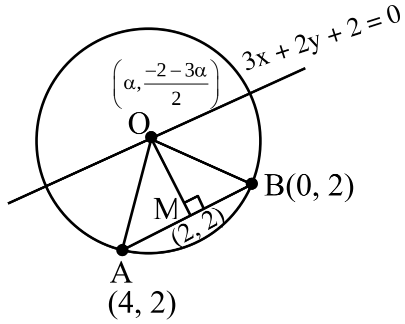
14. Let a circle C pass through the points (4, 2) and (0, 2), and its centre lie on 3x + 2y + 2 = 0. Then the length of the chord, of the circle C, whose mid-point is (1, 2), is:

(1) $\sqrt{3}$ (2) $2\sqrt{3}$

(3) $4\sqrt{2}$ (4) $2\sqrt{2}$

Ans. **(2)**

Sol.



$M_{AB} = 0 \Rightarrow$ OM is vertical

$\Rightarrow \alpha = 2$

$\therefore$ Centre $(0) \equiv (2, -4)$

$r = OA = \sqrt{(2-4)^2 + (2+4)^2} = \sqrt{40}$

mid point of chord is $N \equiv (1, 2)$ $\therefore ON = \sqrt{37}$

$\therefore$ length of chord $= 2\sqrt{r^2 - (ON)^2}$

$= 2\sqrt{40 - 37} = 2\sqrt{3}$

15. Let $A = [a_{ij}]$ be a 2 × 2 matrix such that $a_{ij} \in \{0, 1\}$ for all i and j. Let the random variable X denote the possible values of the determinant of the matrix A. Then, the variance of X is:

(1) $\frac{1}{4}$ (2) $\frac{3}{8}$

(3) $\frac{5}{8}$ (4) $\frac{3}{4}$

Ans. **(2)**

Sol. $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$= a_{11}a_{22} - a_{21}a_{12}$

$= \{-1, 0, 1\}$

x	P_i	P_iX_i	$P_iX_i^2$
–1	$\frac{3}{16}$	$-\frac{3}{16}$	$\frac{3}{16}$
0	$\frac{10}{16}$	0	0
1	$\frac{3}{16}$	$\frac{3}{16}$	$\frac{3}{16}$
		$\sum P_iX_i = 0$	$\sum P_iX_i^2 = \frac{3}{8}$

$\therefore var(x) = \sum P_iX_i^2 - (\sum P_iX_i)^2$

$= \frac{3}{8} - 0 = \frac{3}{8}$

16. Bag 1 contains 4 white balls and 5 black balls, and Bag 2 contains n white balls and 3 black balls. One ball is drawn randomly from Bag 1 and transferred to Bag 2. A ball is then drawn randomly from Bag 2. If the probability, that the ball drawn is white, is 29/45, then n is equal to:

(1) 3 (2) 4

(3) 5 (4) 6

Ans. **(4)**

Sol. Bag 1 = {4W, 5B}

Bag 2 = {nW, 3B}

$P\left(\frac{W}{Bag\,2}\right)=\frac{29}{45}$

$\Rightarrow P\left(\frac{W}{B_1}\right)\times P\left(\frac{W}{B_2}\right)+P\left(\frac{B}{B_1}\right)\times P\left(\frac{W}{B_2}\right)=\frac{29}{45}$

$\frac{4}{9}\times\frac{n+1}{n+4}+\frac{5}{9}\times\frac{n}{n+4}=\frac{29}{45}$

$\boxed{n=6}$

17. The remainder, when 7^{103} is divided by 23, is equal to:

(1) 14 (2) 9

(3) 17 (4) 6

Ans. **(1)**

Sol. $7^{103} = 7\,(7^{102}) = 7\,(343)^{34} = 7\,(345-2)^{34}$

$7^{103} = 23K_1 + 7.2^{34}$

Now $7.2^{34} = 7\,.\,2^2\,.\,2^{32}$

$= 28\,.\,(256)^4$

$= 28\,(253+3)^4$

$\therefore\ 28\times 81 \Rightarrow (23+5)\,(69+12)$

$23K_2 + 60$

$\therefore$ Remainder = $\boxed{14}$

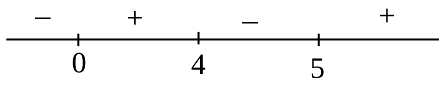
18. Let $f(x)=\int_0^x t\left(t^2-9t+20\right)dt,\ \ 1\le x\le 5$. If the range of f is [α, β], then 4(α + β) equals:

(1) 157 (2) 253

(3) 125 (4) 154

Ans. **(1)**

Sol. $f'(x) = x^3 - 9x^2 + 20x = x(x-4)\,(x-5)$



$\therefore$ **f(x)** = $\frac{x^4}{4}-\frac{9x^3}{3}+\frac{20x^2}{2}$

$f(1)=\frac{1}{4}-3+10=\frac{29}{4}=\alpha$

$f(4)=\frac{256}{4}-3(64)+10(16)=32=\beta$

$4(\alpha+\beta)=4\left(\frac{29}{4}+32\right)=157$

19. Let $\hat{a}$ be a unit vector perpendicular to the vectors $\vec{b}=\hat{i}-2\hat{j}+3\hat{k}$ and $\vec{c}=2\hat{i}+3\hat{j}-\hat{k}$, and makes an angle of $\cos^{-1}\left(-\frac{1}{3}\right)$ with the vector $\hat{i}+\hat{j}+\hat{k}$. If $\hat{a}$ makes an angle of $\frac{\pi}{3}$ with the vector $\hat{i}+\alpha\hat{j}+\hat{k}$, then the value of α is :

(1) $-\sqrt{3}$ (2) $\sqrt{6}$

(3) $-\sqrt{6}$ (4) $\sqrt{3}$

Ans. **(3)**

Sol. Let $\vec{v}=\hat{i}+\hat{j}+\hat{k}$

$$\vec{b}\times\vec{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -2 & 3\\ 2 & 3 & -1\end{vmatrix}$$

$=-7\hat{i}+7\hat{j}+7\hat{k}$

$=-7(\hat{i}-\hat{j}-\hat{k})$

Now $\hat{a}=\frac{\hat{i}-\hat{j}-\hat{k}}{\sqrt{3}}$ or $\frac{-\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$

↓ ↓

$\cos\theta=\frac{\hat{a}.\vec{v}}{|\vec{v}|}=\frac{1-1-1}{\sqrt{3}\sqrt{3}}=\frac{-1}{3}$ $\cos\theta=\frac{\hat{a}.\vec{v}}{|\vec{v}|}=\frac{-1+1+1}{3}=\frac{1}{3}$

(rejected)

$\Rightarrow \hat{a}=\frac{\hat{i}-\hat{j}-\hat{k}}{\sqrt{3}}$

Now $\cos\frac{\pi}{3}=\frac{\hat{a}\cdot(\hat{i}+\alpha\hat{j}+\hat{k})}{\sqrt{1+\alpha^2+1}}$

$\Rightarrow \frac{1}{2}=\frac{1-\alpha-1}{\sqrt{3}\sqrt{\alpha^2+2}}$

$\Rightarrow \frac{\sqrt{3}}{2}\sqrt{\alpha^2+2}=-\alpha \quad (\because\ \alpha<0)$

$3\alpha^2 + 6 = 4\alpha^2$

$\Rightarrow \alpha = -\sqrt{6}$

20. If for the solution curve y = f(x) of the differential equation $\frac{dy}{dx}+(\tan x)y=\frac{2+\sec x}{(1+2\sec x)^2}$, $x\in\left(\frac{-\pi}{2},\frac{\pi}{2}\right), f\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{10}$, then $f\left(\frac{\pi}{4}\right)$ is equal to:

(1) $\frac{9\sqrt{3}+3}{10(4+\sqrt{3})}$ (2) $\frac{\sqrt{3}+1}{10(4+\sqrt{3})}$

(3) $\frac{5-\sqrt{3}}{2\sqrt{2}}$ (4) $\frac{4-\sqrt{2}}{14}$

Ans. (4)

Sol. If $e^{\int \tan x\,dx}=e^{\ell n(\sec x)}=\sec x$

$\therefore$ **y** . sec x = $\int\left\{\frac{2+\sec x}{(1+2\sec x)^2}\right\}$ sec x dx

$=\int\frac{2\cos x+1}{(\cos x+2)^2}dx$ Let $\cos x=\frac{1-t^2}{1+t^2}$

$=\int\frac{2\left(\frac{1-t^2}{1+t^2}\right)+1}{\left(\frac{1-t^2}{1+t^2}+2\right)^2}2dt$

$=\int\frac{2-2t^2+1+t^2}{(1-t^2+2+2t^2)^2}\times 2dt$

$=2\int\frac{3-t^2}{(t^2+3)^2}dt$

Let $t+\frac{3}{t}=u$

$\left(1-\frac{3}{t^2}\right)dt=du$

$=-2\int\frac{du}{u^2}$

y . (sec x) = $\frac{2}{u}$ + c

$$\boxed{y.\sec x=\frac{2}{t+\frac{3}{t}}+c} \quad ....(I)$$

At $x=\frac{\pi}{3}$, $t=\tan\frac{x}{2}=\frac{1}{\sqrt{3}}$

$2.\frac{\sqrt{3}}{10}=\frac{2}{\frac{1}{\sqrt{3}}+3\sqrt{3}}+c$

$2.\frac{\sqrt{3}}{10}=\frac{2\sqrt{3}}{10}+c \Rightarrow C=0$

At $x=\frac{\pi}{4}$, $t=\tan\frac{x}{2}=\sqrt{2}-1$

$\therefore\ y.\sqrt{2}=\frac{2}{\sqrt{2}-1+\frac{3}{\sqrt{2}-1}}$

$y.\sqrt{2}=\frac{2(\sqrt{2}-1)}{6-2\sqrt{2}}$

$y=\frac{\sqrt{2}(\sqrt{2}-1)}{2(3-\sqrt{2})}=\frac{1}{\sqrt{2}}\times\frac{2\sqrt{2}-1}{7}$

$=\frac{4-\sqrt{2}}{14}$

SECTION-B

21. If $24\int_0^{\frac{\pi}{4}}\left(\sin\left|4x-\frac{\pi}{12}\right|+[2\sin x]\right)dx = 2\pi+\alpha$, where [·] denotes the greatest integer function, then α is equal to ________.

Ans. (12)

Sol. $=24\int_0^{\frac{\pi}{48}}-\sin\left(4x-\frac{\pi}{12}\right)+\int_{\pi/48}^{\pi/4}\sin\left(4x-\frac{\pi}{12}\right)$

$+\int_0^{\frac{\pi}{6}}[0]dx+\int_{\pi/6}^{\pi/4}[2\underset{1}{\underset{\downarrow}{\sin}}x]dx$

$=24\left[\frac{\left(1-\cos\frac{\pi}{12}\right)}{4}-\frac{\left(-\cos\frac{\pi}{12}-1\right)}{4}\right]+\frac{\pi}{4}-\frac{\pi}{6}$

$=24\left(\frac{1}{2}+\frac{\pi}{12}\right)=2\pi+12$

$\alpha = 12$

22. If $\lim_{t\to 0}\left(\int_0^1 (3x+5)^t\,dx\right)^{\frac{1}{t}} = \frac{\alpha}{5e}\left(\frac{8}{5}\right)^{\frac{2}{3}}$, then α is equal to ________.

Ans. **(64)**

Sol. 1^∞ form

Now $L = e^{\lim_{t\to 0}\frac{1}{t}\left(\left.\frac{(3x+5)^{t+1}}{3(t+1)}\right|_0^1 - 1\right)}$

$= e^{\lim_{t\to 0}\frac{8^{t+1}-5^{t+1}-3t-3}{3t(t+1)}}$

$= e^{\frac{8\ell n8 - 5\ell n5 - 3}{3}}$

$= \left(\frac{8}{5}\right)^{2/3}\left(\frac{64}{5}\right) = \frac{\alpha}{5e}\left(\frac{8}{5}\right)^{2/3}$

On comparing

$\alpha = 64$

23. Let $a_1, a_2, \ldots, a_{2024}$ be an Arithmetic Progression such that $a_1 + (a_5 + a_{10} + a_{15} + \ldots + a_{2020}) + a_{2024} = 2233$. Then $a_1 + a_2 + a_3 + \ldots + a_{2024}$ is equal to ________.

Ans. **(11132)**

Sol. $a_1 + a_5 + a_{10} + \ldots + a_{2020} + a_{2024} = 2233$

In an A.P. the sum of terms equidistant from ends is equal.

$a_1 + a_{2024} = a_5 + a_{2020} = a_{10} + a_{2015}$ .....

$\Rightarrow$ 203 pairs

$\Rightarrow 203(a_1 + a_{2024}) = 2233$

Hence,

$S_{2024} = \frac{2024}{2}(a_1 + a_{2024})$

$= 1012 \times 11$

$= 11132$

24. Let integers $a, b \in [-3, 3]$ be such that $a + b \neq 0$. Then the number of all possible ordered pairs (a, b), for which $\left|\frac{z-a}{z+b}\right| = 1$ and $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 1$, $z \in C$, where ω and ω^2 are the roots of $x^2 + x + 1 = 0$, is equal to ________.

Ans. **(10)**

Sol. $a, b \in I, -3 \le a, b \le 3, a + b \neq 0$

$|z - a| = |z + b|$

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 1$$

$$\Rightarrow \begin{vmatrix} z & z & z \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 1$$

$$\Rightarrow z\begin{vmatrix} 1 & 1 & 1 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 1$$

$$\Rightarrow z\begin{vmatrix} 1 & 0 & 0 \\ \omega & z+\omega^2-\omega & 1-\omega \\ \omega^2 & 1-\omega^2 & z+\omega-\omega^2 \end{vmatrix} = 1$$

$\Rightarrow z^3 = 1$

$\Rightarrow z = \omega, \omega^2, 1$

Now

$|1-a| = |1+b|$

$\Rightarrow$ 10 pairs

25. Let $y^2 = 12x$ the parabola and S be its focus. Let PQ be a focal chord of the parabola such that (SP) (SQ) $= \frac{147}{4}$. Let C be the circle described taking PQ as a diameter. If the equation of a circle C is $64x^2 + 64y^2 - \alpha x - 64\sqrt{3}\,y = \beta$, then $\beta - \alpha$ is equal to ________.

Ans. **(1328)**

Sol. $y^2 = 12x \quad a = 3 \quad SP \times SQ = \frac{147}{4}$

Let $P(3t^2, 6t)$ and $t_1t_2 = -1$

(ends of focal chord)

So, $Q\left(\frac{3}{t^2}, \frac{-6}{t}\right)$

$S(3, 0)$

$SP \times SQ = PM_1 \times QM_2$

(dist. from directrix)

$= (3 + 3t^2)\left(3+\frac{3}{t^2}\right) = \frac{147}{4}$

$\Rightarrow \frac{(1+t^2)^2}{t^2} = \frac{49}{12}$

$t^2 = \frac{3}{4}, \frac{4}{3}$

$t = \pm\frac{\sqrt{3}}{2}, \pm\frac{2}{\sqrt{3}}$

considering $t = \frac{-\sqrt{3}}{2}$

$P\left(\frac{9}{4}, -3\sqrt{3}\right)$ and $Q\left(4, 4\sqrt{3}\right)$

Hence, diametric circle:

$(x-4)\left(x-\frac{9}{4}\right) + (y+3\sqrt{3})(y-4\sqrt{3}) = 0$

$\Rightarrow x^2 + y^2 - \frac{25}{4}x - \sqrt{3}y - 27 = 0$

$\Rightarrow \alpha = 400, \beta = 1728$

$\beta - \alpha = 1328$